Mathematics 433 **Fall 2010** **Turn In Set: 8**

November 23, 2010

____
Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**
"*A life spent making mistakes is not only more honorable, but more useful than a life spent doing nothing.*"
– George Bernard Shaw

You may submit at most four (4) **of these problems.**

1. Do both of the following.
   (a) Show that the multiplicative group of nth roots of unity is isomorphic to $\mathbf{Z}_n$.
   (b) Let $G = \mathbf{R}\setminus\{-1\}$ and define a binary operation on G by $a * b = a + b + ab$.
   Prove that G is a group under this operation. Also show that G is isomorphic to the multiplicative group of nonzero real numbers.
2. Prove that D_4 cannot be isomorphic to the internal direct product of two of its proper subgroups.
3. Prove that $S_3 \times \mathbf{Z}_2$ is isomorphic to D_6. Make a conjecture about D_n and prove your conjecture.
4. Do both of the following.
   (a) Prove or disprove. Every abelian group of order divisible by 3 contains a subgroup of order 3.
   (b) Prove or disprove. Every nonabelian group of order divisible by 6 contains a subgroup of order 6.
5. Let G be a group of order 20. If G has subgroups H and K of orders 4 and 5, respectively and $hk = kh$ for every $h \in H$ and $k \in K$, prove that G is the internal direct product of H and K.
   (a) Do two of the following.
   Prove or disprove. Let G, H, K be groups. If $G \times H \cong G \times K$, then $H \cong K$.
   (b) Prove or disprove. There is a noncyclic abelian group of order 51.
   (c) Prove or disprove. There is a noncyclic abelian group of order 52.
6. Do either of the following.
   (a) Prove that S_n is isomorphic to a subgroup of A_{n+2}.
   (b) Prove that D_n is isomorphic to a subgroup of S_n.
7. Recall that if X is any set then the set of all permutations $\pi : X \to X$ is a group S_X under function composition. Recall also that if G is a group then the set $Aut\,(G)$ of all automorphisms of G is a subgroup of S_G.
   (a) Find $Aut\,(\mathbf{Z}_6)$
   (b) Find $Aut\,(\mathbf{Z})$

8. Recall that if G is a group and if $g \in G$ then the map ϕ_g defined by $\phi_g(x) = gxg^{-1}$ is an automorphism of G. We call the set of all such conjucation maps the set of **inner** automorphism of G, $Inn(G) = \left\{\phi_g : \phi_g(x) = gxg^{-1} \text{ for all } x \in G\right\}$.

   (a) Find two nonisomorphic groups H and G for which $Aut(G) \cong Aut(H)$

   (b) Prove that $Inn(G)$ is a subgroup of $Aut(G)$.

9. Do **one** of the following.

   (a) Let $n_1, \cdots, n_k$ be positive integers. Show that

   $$\prod_{i=1}^{k} \mathbf{Z}_{n_i} \cong \mathbf{Z}_{n_1 \cdots n_k}$$

   if and only if $\gcd(n_i, n_j) = 1$ for $i \neq j$.

   (b) If G is the internal direct product of $H_1, \cdots, H_n$, prove that G is isomorphic to

   $$\prod_{i=1}^{n} H_i$$