Semester Review for MATH 180

The Big Picture:

Chapter 1: Presents the basics of functions, graphs and a review of pertinent algebra

Chapter 2: Limits, Continuity, Vertical Asymptotes

Chapter 3: Introduction to Differential Calculus

Chapter 4: Applications of the Differential Calculus and L'Hôpital's Rule

Chapter 5: Basics of Integral Calculus

More Detailed Outline

Chapter 1 Preliminary Algebraic Information

- Preliminary algebra

  1. Absolute Value definition:
  $$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$
  2. Distance on the line and in the plane
     (a) $|x - a|$
     (b) $\sqrt{(x-a)^2 + (y-b)^2}$
  3. Interval notation: $|x - a| < b$ is the set of all points in the open interval $(a - b, a + b)$
  4. Graph of an equation
     (a) The set of points (x, y) making the equation true.
  5. Equation of circle centered at the point (h, k) and of radius R: $(x-h)^2 + (y-k)^2 = R^2$
  6. Basic Trigonometric Functions
     (a) Exact Trigonometric Values for $\theta = 0,\ \pi/6,\ \pi/4,\ \pi/3,\ \pi/2,\ \pi$
     (b) Periods
     (c) Trigonometric Identities

- Equations of lines in the plane

  1. Slope
  2. Point-Slope form
  3. Slope-intercept form
  4. Standard form
  5. Vertical, Horizontal lines

- Basics of functions and their graphs

1. A **function** is a rule that assigns to each element x of a set D a unique element $y = f(x)$ of a set Y. The element y is called the **image** of x under f and is denote by $f(x)$. The set D is called the **domain** of f, and the set of all images of elements of X is called the range of the function f. The set Y is called the **codomain** of the function f.

2. Piecewise defined functions

$$f(x) = \begin{cases} x^2 + 1 \text{ if } x \geq 2 \\ -7 \text{ if } x < 0 \end{cases}$$

3. Equality of functions: Two functions f and g are said to be equal (written $f = g$) if and only if
   (a) f and g have the same domain and
   (b) $f(x) = g(x)$ for every x in the domain.

4. The sum, difference, product, quotient and scaling of functions
   (a) $(f \pm g)(x) = f(x) \pm g(x)$ (Domain is $Dom(f) \cap Dom(g)$)
   (b) $(fg)(x) = f(x)\, g(x)$ $(Dom(f) \cap Dom(g))$
   (c) $(f/g)(x) = f(x)/g(x)$ $(Dom(f) \cap Dom(g)$ and $g(x) \neq 0)$
   (d) $(cf)(x) = c \cdot f(x)$ where c is a constant.

5. Composition of functions. The **composite** function $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x))$$

   for each x in the domain of g for which $g(x)$ is in the domain of f.

6. The **graph** of a function f is the set of points (x, y) that satisfy the equation $y = f(x)$ for all x in the domain of f.

7. Scaling and Shifting a graph: $y = f(x)$ versus $y - k = af(b(x - h))$
   (a) "stretches" $y = f(x)$ vertically by a factor of a, "compresses" the result horizontally by a factor of b then shifts that result horizontally by h and vertically by k.

8. Vertical line test for whether a curve in the plane is the graph of a function.

9. Intercepts of graphs.

10. Two variables x and y are **proportional** if there is a nonzero constant k with $y = kx$.

11. Even and Odd functions
    (a) A function f is **even** if $f(-x) = f(x)$ for every x in the domain of f.
        i. The graph of an even function is symmetric with respect to the y axis.
    (b) A function f is **odd** if $f(-x) = -f(x)$ for every x in the domain of f.
        i. The graph of an odd function is symmetric with respect to the origin.

12. A list of basic functions
    (a) constant: $f(x) = a$
    (b) linear: $f(x) = mx + b$
    (c) power: $f(x) = x^a$, where a is a constant
    (d) quadratic: $f(x) = ax^2 + bx + c$

(e) cubic: $f(x) = ax^3 + bx^2 + cx + d$

(f) polynomial: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

(g) rational:

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials.

(h) greatest integer function $f(x) = \lfloor x \rfloor$ outputs the greatest integer less than or equal to x.

(i) exponential functions $f(x) = a^x$ where $a > 0$ is a constant and $a \neq 1$

   i. Special case $f(x) = e^x$

(j) logarithmic functiosn $f(x) = \log_a(x)$ (the inverse function to $f(x) = a^x$)

   i. Special case $f(x) = \ln(x)$

- Inverse functions in general and inverse trigonometric function

1. A function f with domain D and range R whose graph which is one-to-one (satisfies the horizontal line test) is said to have an inverse function f^{-1}.
2. Such a function satisfies both
   (a) $f^{-1}(f(x)) = x$ for all x in the set D
   (b) $f(f^{-1}(y)) = y$ for all y in the set R
3. The domain of f^{-1} is the range of f and vice versa.
4. The graph of f^{-1} is the reflection of the graph of f across the line $y = x$.
5. If we restrict the domains of the trigonometric functions appropriately, then the resulting restricted functions have inverses.
   (a) $\arcsin(x)$, $\arctan(x)$, $\text{arcsec}(x)$, $\arccos(x)$, $\text{arccot}(x)$, $\text{arccsc}(x)$

- Graphing with calculators or computers requires careful choice of the "window".

Chapter 2

Limits: The real basis of calculus

- Intuition – what a function "ought to be" at a point.

1. Any limit that is not in an "indeterminate form" (see L'Hôpital's Rule below) can easily be evaluated **informally**. This is because most such limits are associated with points of continuity of functions and hence those functions behave the way they "ought to".
   (a) A limit that has an "indeterminate form" must be informally evaluated in a different manner.
2. All limits can be evaluated **formally.** This involves using the $\varepsilon - \delta$ definition and writing a proof of the value of the limit. Usually, the argument is done backwards as scratchwork then presented in the form of a logical deduction.

(a) For example: $\lim_{x\to 1/2} \frac{4x^2-1}{2x-1} = 2$ is true because
If ε is any positive number then we can
choose $\delta = \frac{1}{2}\varepsilon$ and then
whenever $0 < |x - 1/2| < \delta$
we have $|x - 1/2| < \frac{1}{2}\varepsilon$ and $x \neq \frac{1}{2}$
$\left|\frac{2x-1}{2}\right| < \frac{1}{2}\varepsilon$ and $x \neq \frac{1}{2}$
$|2x - 1| < \varepsilon$ and $x \neq \frac{1}{2}$
$\left|\frac{(2x-1)^2}{2x-1}\right| < \varepsilon$ and $x \neq \frac{1}{2}$
$\left|\frac{4x^2-1-4x+2}{2x-1}\right| < \varepsilon$ and $x \neq \frac{1}{2}$
$\left|\frac{4x^2-1}{2x-1} - 2\right| < \varepsilon$

- **Definition:** When we write $\lim_{x\to a} f(x) = L$ we mean the following statement is true.

  1. Given any positive number ε (which defines a horizontal band of width 2ε centered at height L on the graph of $y = f(x)$), it is possible to find a positive number δ (which defines a vertical band of width 2δ centered at $x = a$) satisfying the following.

     Whenever x is a number where $0 < |x - a| < \delta$ (that is, $x \neq a$ is in the vertical band mentioned above) then $|f(x) - L| < \varepsilon$ (that is, $f(x)$ is in the horizontal band mentioned above).
  2. Note that when this definition is true, then for every x other than a, the graph of $y = f(x)$ enters the rectangle formed by the two bands from the left and exits from the right (not the top or bottom).

- Not all limits exist.

  1. A limit exists if and only if the corresponding Left-hand limit and Right-hand limit both exist.

- Algebraic manipulation of limits

  1. Limits behave we would like them to with respect to addition, subtraction, multiplication and division. For example, the limit of a product of functions is the product of the limits of the functions provided all the limits involved exist. For example we have a theorem that proves $\lim_{x\to a} f(x) g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
  2. This allows us to informally evaluate more complex limits by breaking them down into sums, products, etc. of simpler limits.
  3. The Sandwich Theorem is useful for some difficult to compute limits. We used it to show that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

- Continuity: functions that "are what they ought to be"

  1. A function f is continuous at the number c if
     (a) c is in the domain of f
     (b) $\lim_{x\to c} f(x)$ exists
     (c) $\lim_{x\to c} f(x) = f(c)$

2. Functions that are built up by adding, subtracting, multiplying, dividing, or composing continuous functions are also continuous.
3. Continuous functions are central to the study of calculus because they behave the way they "ought to" with respect to limits.

- Exponential and Logarithmic functions

1. The exponential and logarithmic functions are inverse functions.
   (a) $e^{\ln(x)} = x$ and $\ln(e^y) = y$ for all x in the domain of $f(x) = \ln(x)$ - that is all $x > 0$ and all y in the domain of $g(y) = e^y$ - that is $(-\infty, \infty)$
2. They are continuous and are used in many mathematical models.

- The graph of a function has a vertical asymptote at the number $x = a$ if and only if either the Left-hand or Right-hand limit is infinite.
- Indeterminate forms are "$\frac{0}{0}$" and any other form that can be converted into "$\frac{0}{0}$" . For example,

1. "$\frac{\infty}{\infty}$" converts to "$\frac{\frac{1}{\infty}}{\frac{1}{\infty}}$" which is "$\frac{0}{0}$",
2. "$0 \cdot \infty$" converts to "$\frac{0}{\frac{1}{\infty}}$",
3. "$\infty - \infty$" factors to "$0 \cdot \infty$"
4. Also, by taking logarithms we can convert "1^∞", "∞^0", and "0^0" to "$\infty \cdot 0$", "$0 \cdot \infty$", and "$0 \cdot -\infty$", respectively. These can then be converted, as above, into the canonical "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" forms.

Chapter 3

Introduction to Differential Calculus

- Graphical Interpretation: the derivative of a function at c is the slope of the tangent line to the graph of the function at the point $(c, f(c))$.

1. A function with a derivative at c looks like a line (the tangent line) when we zoom in on the graph near the point $(c, f(c))$.

- **Definition**:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Differentiability implies continuity: If you can take the derivative of a function at the number a then that function is continous at the number a.

1. Intuition If the graph of a function looks like a line on both sides of the input a then the function will be continuous at a.

- Rules and Formulas for derivatives (How to take derivatives of almost any function)

1. Basic Rules: Power Rule, Constant multiple rule, sum rule, difference rule, linearity rule, product rule, quotient rule.
2. Trigonometric, inverse trigonometric, exponential and logarithmic formulas
   (a) For example: $\frac{d}{dx}[\sin(x)] = \cos(x)$, $\frac{d}{dx}\left[\arctan(x) = \frac{1}{1+x^2}\right]$
3. Chain Rule $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$
   (a) The most important derivative rule.

- Parametric equations: express the points (x, y) that lie on a curve C using separate functions for x and y.

1. Example: $x(t) = 2\cos(t)$, $y(t) = 9\sin(t)$, $0 \le t \le 2\pi$ is a parametric form of describing the graph of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

- Rates of Change – as applications of derivatives

1. Mathematical Modeling
   (a) For example: rectilinear motion
   (b) velocity is the derivative of position and acceleration is the derivative of velocity
      i. $v(t) = s'(t)$
      ii. $a(t) = v'(t) = s''(t)$
2. Relative rate of change

$$\frac{f'(x)}{f(x)}$$

3. Percentage rate of change is the relative rate of change expressed as a percentage.

- Implicit Differentiation

1. Take derivatives of functions without first solving for the function.
2. **Example:** $\cos(x+y) + y = 2$ tells us that

$$\begin{aligned} \frac{d}{dx}[\cos(x+y)+y] &= \frac{d}{dx}[2] \\ -\sin(x+y)\left(1+\frac{dy}{dx}\right)+\frac{dy}{dx} &= 0 \\ (-\sin(x+y)+1)\frac{dy}{dx} &= \sin(x+y) \\ \frac{dy}{dx} &= \frac{\sin(x+y)}{-\sin(x+y)+1} \end{aligned}$$

- Related rates of change – more applications of derivatives.

1. Many physical situations involve the rate at which two quantities are changing where the rate of change of one quantity determines the rate of change of the other.

2. In these situations, determine which quantities are changing, draw a figure illustrating the quantities, name them with variables, determine a formula or equation relating the quantities, use implicit differentiation to compute the derivatives, and answer the question that is asked.

- Linear approximation and differentials

1. The tangent line to a graph is almost the same as the graph of the function

$$\begin{aligned} f(x) &\approx L(x) = f(a) + f'(a)(x-a) \\ f(x) - f(a) &\approx f'(a)(x-a) \\ \Delta f &\approx f'(a)\Delta x \\ \Delta f &\approx df \end{aligned}$$

2. Error in measurement: $\Delta x = (x + \Delta x) - x$ (exact value minus measured value)
3. Propagated error: $\Delta f = f(x + \Delta x) - f(x)$
4. Relative error: $\frac{\Delta f}{f} \approx \frac{df}{f}$
5. and Percentage error: $100\left(\frac{\Delta f}{f}\right)\%$

- Hyperbolic functions:

1. $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$

Chapter 4: Applications of the Derivative

- Extreme Value Theorem

1. Absolute (global) maxima and minima can only occur at:
   (a) endpoints
   (b) where $f\,'$ DNE or
   (c) where $f\,'(x) = 0$
2. Relative (local) maxima and minima
   (a) (can only occur inside an open interval of the domain)
   (b) where $f\,'$ DNE or
   (c) where $f\,'(x) = 0$
   (d) Never at an endpoint

- Rolle's Theorem: If f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$. Then there is at least one number c in (a, b) at which $f\,'(c) = 0$.

- Mean Value Theorem: If f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$. Then there is at least one number c in (a, b) at which $f\,'(c) = \frac{f(b)-f(a)}{b-a}$.

1. This allows us to prove both of the following.

(a) If $f'(x) = 0$ for all x in an interval then $f(x)$ is constant on that interval

2. Constant Difference Theorem

   (a) If $f'(x) = g'(x)$ for all x in an open interval then they differ by a constant on that interval. That is, $g(x) = f(x) + C$

- Sketching graphs

  1. Critical points: where $f'(x)$ DNE or equals 0
  2. Increasing/Decreasing: intervals where $f'(x)$ is either positive or negative.
  3. Inflection points: $f''(x)$ changes sign (and there is a tangent line)
  4. Concave up/down: intervals where $f''(x)$ is either positive or negative.
  5. First Derivative Test for local extrema
  6. Second Derivative Test for local extrema

- Sketching graphs and including asymptotes and vertical tangents

  – Horizontal Asymptotes

    1. the horizontal line $y = L$ if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$

  – Vertical Asymptotes

    1. The vertical line $x = a$ if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$

  – Vertical tangents and cusps

    1. A vertical tangent or cusp at the number a if $\lim_{x\to a^+} f'(x) = \pm\infty$ or $\lim_{x\to a^-} f'(x) = \pm\infty$

  – An oblique asympote of $y = mx + b$ if $\lim_{x\to\pm\infty} \frac{f(x)}{mx+b} = 1$.

- L'Hôpital's Rule and indeterminate forms

  1. Only works for "$\frac{0}{0}$" and "$\frac{\pm\infty}{\pm\infty}$"
  2. For other indeterminate forms use algebra or logarithms to convert into one of the above.

     (a) "$0 \cdot \infty$" converts to "$\frac{0}{\frac{1}{\infty}}$"

     (b) "$\infty - \infty$" can factor to "$0 \cdot \infty$"

     (c) "1^∞" converts by using logarithms to "$\infty \cdot 0$" which converts to "$\frac{0}{\frac{1}{\infty}}$"

     (d) "∞^0" converts by using logarithms to "$0 \cdot \infty$" which converts to "$\frac{0}{\frac{1}{\infty}}$"

     (e) "0^0" converts by using logarithms to "$0 \cdot -\infty$" which converts to "$\frac{0}{\frac{1}{-\infty}}$"

- Optimization in Physical Sciences as well as Business, Economics and the Life Sciences

  1. Draw a figure and label appropriate quantities
  2. Determine what is to be maximized or minimized and with respect to what quantity
  3. Express the quantity to be optimized as a function of a single variable

4. Find the domain of this function.
5. Find the optimum

- Newton-Raphson Method for approximating zeros of functions.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

1. Usually works when first guess is "close" to a zero of the function.
2. Converges rapidly when it works.
3. Can use Bisection Method when Newton's Method does not converge.

Chapter 5: Integration

- Antidifferentiation

1. The reverse of taking a derivative
2. If $F'(x) = G'(x)$ then $G(x) = F(x) + C$
3. Slope fields for graphing antiderivatives
4. Rules and formulas for antiderivatives (reverse the derivative formulas)
5. Area as an antiderivative

- Areas as limit of a sum

1. Sigma notation and finding areas "the hard way".
2. Approximate the area using a Riemann sum with n subintervals
3. Rewrite the sum in a form where you can use sigma notation to simplify
4. Take the limit as n goes to infinity to find the exact area.

- Riemann Sums and definite integrals:

$$\sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

1. Using sums of linear approximations over small intervals to approximate effects of functions over large intervals.
2. A Riemann Sum depends on
   (a) the function $f(x)$
   (b) an interval $[a, b]$ in the domain of f
   (c) a partition $P:\ a = x_0 < x_1 < \cdots < x_n = b$ of the interval
   (d) a selection of points $c_1,\ c_2, \cdots, c_n$ where c_k is a point in the k'th subinterval $[x_{k-1}, x_k]$ of the partition.

3. A definite integral is the limit as the partition norm goes to 0 of all possible Riemann sums for a function f on the interval $[a, b]$

$$\int_a^b f(x)\ dx = \lim_{||P \to 0||} \sum_{k=1}^{n} f(c_k)\, \Delta x_k$$

- Fundamental Theorems of Calculus

1. Fundamental Theorem of Calculus – Part 1: $\frac{d}{dx}\left[\int_a^x f(t)\ \ dt\right] = f(x)\,.$
   (a) Gives us an antiderivative for every continuous function.
   (b) Allows us to compute complex derivatives using the chain rule

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\ \ dt\right] = f(g(x))\, g'(x)$$

2. Fundamental Theorem of Calculus – Part 2: $\int_a^b f(x)\ \ dx = F(b) - F(a)$ where $F'(x) = f(x)\,.$
   (a) Shows us how to "easily" compute definite integrals (without using limits of Riemann Sums).
   (b) Requires that you know an antiderivative of the given function.