Semester Review

The Big Picture:

Chapter 5: Presents the basics of the theory of integration

- Accumulation via Riemann Sums

Chapter 6: Standard applications of definite integrals

Chapter 7: Finding antiderivatives, approximations, and Improper Integrals

Chapter 8: Sequences and Series

- Sequences and Series are different
- How to determine convergence.
- Every power series is a function with a special domain.
- Some functions are equal to power series (their Taylor Series).

The Medium Picture:

Chapter 5

The basic theory of integration

- Antiderivatives (From first semester calculus)
  1. Indefinite Integral notation
- Functions with Interval Domains
  1. Estimating using finite sums
  2. Sigma notation and limits of finite sums
  3. Riemann Sums, limits, and definite integrals
  4. Fundamental Theorems of Calculus
  5. Basic Substitution techniques (Rule of Thumb)
  6. Area between curves as an integral.

Chapter 6

Standard Applications of Definite Integrals

- 1. General Process of building a formula using Riemann Sums
  2. Areas between curves (from Chapter 5)
  3. Volumes of solids
     (a) Slicing
     (b) Rotation about an axis
  4. Arc length of curves in the plane
  5. Separable Differential Equations and Exponential Change

Chapter 7

Methods of Integration

- 1. Integration by parts
  2. Integrals of Trigonometric functions
  3. Computing integrals using Trigonometric Substitutions
  4. Partial Fractions for integrating **any** rational function
  5. Tables of integrals and Computer Algebra Systems
  6. Numerical Integration
  7. Improper integrals
     - Various types
     - Comparison tests to determine the **fact** of convergence.

Chapter 8

Infinite sequences and series

- Sequences
- Infinite Series
  1. **Sequence of partial sums**
- Exact Sums:
  1. Geometric Series
  2. Telescoping Series
- Tests for convergence
  1. Apply to **any** series
     (a) nth term test
     (b) Absolute Convergence Test
        i. Absolute convergence
        ii. Conditional convergence
  2. Apply only to Series with Positive or non-negative terms (or the negatives of such series)
     (a) P-Series
     (b) Comparison Tests (direct and limit)
     (c) Integral Test
        i. Has a bound on error of an estimate
     (d) Ratio and Root Tests
  3. Apply only to series whose terms alternate in sign
     (a) Alternating Series Test
        i. Easy bound on error of an estimate
- Power series:
  1. Every power series is a function with a special type of domain
  2. Differentiation and Integration maintain the center and radius of convergence but not convergence at endpoints
- Taylor Series and Maclaurin Series:
  1. Every function with derivatives of all orders gives rise to an associated power series
- Convergence of Taylor Series:
  1. For what values of x does a function equal its Taylor Series?
- Special functions and their Taylor Series
  1. Binomial Series, e^x, $\sin(x)$, $\cos(x)$, $\frac{1}{1-x}$
  2. Functions obtained from differentiating, integrating, multiplying and adding the above.

More Detailed Outline

Chapter 5: The fundamentals of integration

The basic theory of integration of Functions with Interval Domains

Antiderivatives (antidifferentiation)

- Reversing the process of taking derivatives.
- Harder than differentiation
- Indefinite integral notation

Estimating using finite sums

- Areas, Distance travelled, Displacement
- Any property that can be approximated by **accumulation** of many "smaller" and simpler structures that arise from a "nice" function.

Sigma notation and limits of finite sums

- Changing indices in a finite sum: $\sum_{k=1}^{n} f(k) = \sum_{j=4}^{n+3} f(j-3)$
- Rewriting without first few terms: $\sum_{k=1}^{n} k^2 = 1 + 4 + \sum_{k=3}^{n} k^2 = 1 + 4 + \sum_{j=1}^{n-2} (j+2)^2$

Riemann Sums and definite integrals:

$$\sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

- Using sums of linear approximations over small intervals to approximate effects of functions over large intervals.
- Different Riemann sums can be obtained by varying any of the following
  1. the function $f(x)$
  2. the interval $[a, b]$ in the domain of f
  3. the partition $P: \ a = x_0 < x_1 < \cdots < x_n = b$ of the interval
  4. the selection of points $x_1^*, \ x_2^*, \cdots, x_n^*$ where x_k^* is a point in the k'th subinterval $[x_{k-1}, x_k]$ of the partition.
- A definite integral is the limit, **if it exists**, as the partition norm goes to 0 of all possible Riemann sums for a function f on the interval $[a, b]$

$$\int_a^b f(x)\ dx = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

  1. This limit only exists if it does not matter how one partitions the interval $[a, b]$ nor how one selects the points x_k^* in the subintervals.
  2. This limit **will exist** if the function f is continuous on the interval $[a, b]$. (A result from advanced calculus)

The Fundamental Theorem of Calculus

- How to compute definite integrals without using the limit of Riemann sums.
- Mean Value Theorem for Integrals and **Average Value** of a continuous function
  1. Average of f on $[a, b]$ is

$$\frac{1}{b-a} \int_a^b f(x)\ dx$$

  2. Geometric meaning of the average value: height of rectangle over base $a \leq x \leq b$ with same area as $\int_a^b f(x)\ dx$.

- Fundamental Theorem - Part 1. Every continuous function has an antiderivative. (Proof uses Mean Value Theorem for integrals). This can also be phrased as: Every continuous function is the rate of change of its accumulation function.

$$\begin{aligned} F(x) &= \int_a^x f(t)\ dt \\ F'(x) &= \frac{d}{dx}\int_a^x f(t)\ dt \\ &= f(x) \end{aligned}$$

- Fundamental Theorem – Part 2. Computation of definite integrals (limits of Riemann Sums) can be shortened by the use of antiderivatives (provided one can find a **nice** antiderivative for f) the proof uses part 1 of the FTC.

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Basic Integration techniques

- Substitution using $u = g(x)$

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\ du$$

- Rule of Thumb "substitute for the inside of ugliest thing" usually works for simple integrals

Area Between Curves

- Vertical rectangles gives $\int_a^b [f(x) - g(x)]\ dx$
- Horizontal rectangles gives $\int_c^d [f(y) - g(y)]\ dy$

Chapter 6: Standard Applications of using Riemann Sums

Volumes of solids

Cross-sectional areas give rise to the formula

$$V = \int_a^b A(x)\ dx$$

- Formula applies to any solid with "nice" cross sections.
- **Special Case:** If the solid is obtained as a solid of revolution then cross sections perpendicular to the axis of revolution are particularly "nice".
    1. Disks
    2. Washers

Nesting Cylindrical Shells gives the formula

$$V = 2\pi \int_a^b (\text{shell radius})\,(\text{shell height})\, dt$$

- Formula applies **only** to solids of revolution and where the shells are centered on the axis of revolution.

Arc length and Surface area:

- Use

$$\begin{aligned} ds &= \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt \\ &= \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx \\ &= \sqrt{\left[\frac{dx}{dy}\right]^2 + 1 dy} \end{aligned}$$

in the formulas:

$$\begin{aligned} Length &= \int_a^b ds \\ Area &= \int_a^b 2\pi \left(\text{ribbon radius}\right) \; ds \\ &\text{We did not cover Surface Area.} \end{aligned}$$

- Many problems are 'cooked' so that the algebra simplifies to remove the square root.
- Arc length formula requires curve be differentiable and smooth
- Surface area formula requires the surface be obtained by rotating a smooth curve about an axis.

Exponential Change and Separable Differential Equations:

- Solutions to diffential equations are functions that make the equation true.
  1. $y = Ke^{3x}$ is a solution (for any choice of constant K) to $\frac{dy}{dx} = 3y$ because when $y = Ke^{3x}$ we have $\frac{dy}{dx} = 3Ke^{3x} = 3y$ showing the differential equation holds for this function y.
  2. If a differential equation also has an **initial condition,** $y(0) = y_0$ then a solution must also make that initial condition true.
     (a) For example, $y = 5e^{3x}$ solves $\frac{dy}{dx} = 3y$, where $y(0) = 5$ but $y = 12e^{3x}$ does not.
- Exponential change occurs whenever a quantity changes at a rate proportional to the amount of quantity present. The model is

$$\frac{dy}{dt} = ky$$

  – Examples include:

  1. (a) Radioactive decay
     (b) Population growth
     (c) Continuous interest
     (d) Heat transfer between an object and its surroundings
- Separable differential equations are those that can be written in the form

$$h(y)\frac{dy}{dx} = g(x)$$

  1. To solve, separate the variables and integrate both sides.

Chapter 7: Methods of Integration

Integration by Parts

$$\begin{aligned} \int f(x) g'(x) \, dx &= f(x) g(x) - \int g(x) f'(x) \, dx \\ \int u \, dv &= uv - \int v \, du \end{aligned}$$

- Look for a product of functions, fg', where g' has an "easy" antiderivative, g, and where the product gf' is "easier" to integrate than the original problem.

Integrals of Trigonometric Functions

- Powers of Sine and Cosine

  1. Look for an odd power of either $\sin(x)$ or $\cos(x)$
     (a) substitute u for the other one (e.g. if $\cos(x)$ occurs to an odd power, let $u = \sin(x)$ so that $du = \cos(x)\ dx$)
     (b) Use trigonometric identites to swap out even powers of the non- u trig function.
  2. If both $\sin(x)$ and $\cos(x)$ are to even powers
     (a) Use the half-angle trigonometric identies to reduce to an odd power

$$\begin{aligned}\sin^2(x) &= \frac{1}{2}(1-\cos(2x))\\ \cos^2(x) &= \frac{1}{2}(1+\cos(2x))\\ \sin(2x) &= 2\sin(x)\cos(x)\end{aligned}$$

- Powers of Secant and Tangent (or Cosecant and Cotangent)

  1. Look for an even power of the secant
     (a) substitute for $u = \tan(x)$ so $du = \sec^2(x)\ dx$
     (b) Use trigonometric identites to swap extra even powers of secant for even powers of tangent.
  2. Look for an odd power of the tangent
     (a) substiture $u = \sec(x)$ so $du = \sec(x)\tan(x)\ dx$
     (b) Use trigonometric identites to swap extra even powers of tangent for even powers of secant
  3. If secant is to an odd power and tangent is to an even power (so neither of the first two techniques work), try integration by parts with $dv = \sec^2(x)\,dx$

Trigonometric substitutions

- If $a^2 - u^2$ occurs, try $u = \sin(x)$ or $u = \tanh(x)$
- If $a^2 + u^2$ occurs, try $u = \tan(x)$ or $u = \sinh(x)$
- If $u^2 - a^2$ occurs, try $u = \sec(x)$ or $u = \cosh(x)$

Partial Fractions for integrating rational functions

- Only works on **proper** fractions so **divide first.**
- decompose into sums of fractions with linear, irreducible quadratic, or powers of linear or irreducible quadratic denominators
- Integrate each of the simpler fractions using other techniques

Tables of Integrals and Computer Algebra Systems

- Be very careful when using these.

  1. Can be cumbersome to use
  2. Might have mistakes
  3. Hidden roundoff and theoretical errors in computer implementations

Numerical Integration (Approximating definite integrals with attention to accuracy)

- Left and Right endpoint rules L_n and R_n

  1. Simplest possible techniques to implement but not very efficient. That is, it takes a huge value of n to obtain great accuracy.

- Midpoint Rule: M_n

  1. A bit harder to implement than L_n and R_n but much more efficient.

- Trapezoid Rule: T_n is the average of the Left and Right endpoint rules: $T_n = \frac{L_n + R_n}{2}$
  1. Much more efficient than either L_n or R_n
  2. On the same order of efficiency as M_n.
  3. Error Bound for T_n is: $\left|\int_a^b f(x)\,dx - T_n\right| \le \frac{(b-a)^3}{12n^2} M$
- Simpson's Rule: $S_n = \frac{T_n + 2M_n}{3}$
  1. Exploits the fact that the Trapezoid error tends to be about twice the size of the Midpoint error but opposite in sign.
  2. Error Bound for S_n is: $\left|\int_a^b f(x)\,dx - S_n\right| \le \frac{(b-a)^5}{180n^4} M$

Improper integrals

- 1. Must reduce the problem to a sum of improper integrals with **exactly one** impropriety
  2. Six Types

$$\int_a^\infty f(x)\,dx$$
$$\int_{-\infty}^b f(x)\,dx$$
$$\int_{-\infty}^\infty f(x)\,dx$$
$$\begin{aligned}\int_a^b f(x)\,dx \text{ where } x &= b \text{ is a vertical asymptote}\\ \int_a^b f(x)\,dx \text{ where } x &= a \text{ is a vertical asymptote}\\ \int_a^b f(x)\,dx \text{ where } x &= c \text{ is a vertical asymptote and } a < c < b\end{aligned}$$

  3. These integrals converge if and only if the appropriate limit(s) exist(s).
  4. Direct and Limit Comparison tests
     (a) For when exact evaluation of the integral is not possible.
     (b) Methodology is exactly anaogous to comparison tests for whether or not an infinite series converges.
- Hyperbolic Tirgonometric functions
  1.

$$\begin{aligned}\sinh(x) &= \frac{1}{2}\left(e^x - e^{-x}\right)\\ \cosh(x) &= \frac{1}{2}\left(e^x + e^{-x}\right)\\ \tanh(x) &= \frac{\sinh(x)}{\cosh(x)}, \text{ etc.}\\ \cosh^2(x) - \sinh^2(x) &= 1\end{aligned}$$

  2.

$$\begin{aligned}\frac{d}{dx}[\sinh(x)] &= \cosh(x)\\ \frac{d}{dx}[\cosh(x)] &= \sinh(x)\end{aligned}$$

Chapter 8: Sequences and Series

Sequences

- A sequence is a function with domain the set of positive integers.
- Deduce the general term from a given sequence written in 'dot, dot, dot' form.
- The definition of what it means for a sequence a_n to converge:
  $\lim_{n\to\infty} a_n = L$ means:
  Given any positive number ε, there is a number N for which: if $n > N$ then $|a_n - L| < \varepsilon$
- The Nondecreasing Sequence Theorem for sequences.
  A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.
  1. A sequence a_n is bounded above if there is a number M for which $a_n \le M$ for all n.
  2. A sequence a_n is bounded below if there is a number m for which $m \le a_n$ for all n.
  3. Sequences can be monotone in four ways: increasing, decreasing, nondecreasing, nonincreasing.

Series

- Infinite Series are the discrete analogs of improper integrals of continuous functions.

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx = \lim_{b\to\infty} F(x)\Big|_a^b \qquad \sum_{k=1}^\infty a(k) = \lim_{n\to\infty} \sum_{k=1}^n a(k) = \lim_{n\to\infty} A(k)\big|_1^{n+1}$$

- An infinite series converges if and only if its sequence of partial sums ($s_n = \sum_{k=1}^n a_k$) converges.
- Textbook Notation for infinite series $\sum_{k=1}^\infty a_k$ or $\sum_{n=1}^\infty a_n$.
- Linearity of **convergent** series
  1. If $\sum_{k=1}^\infty a(k)$ and $\sum_{k=1}^\infty b(k)$ both converge then so does $\sum_{k=1}^\infty [r\,a(k) + s\,b(k)]$ where r and s are any constants.
- If r and s are constants – neither equal to 0 then
  1. If any two of $\sum_{k=1}^\infty a(k)$, $\sum_{k=1}^\infty b(k)$,and $\sum_{k=1}^\infty [r\,a(k) + s\,b(k)]$ converge, then so does the third.
- Sums involving **divergent** series
  1. If $\sum_{k=1}^\infty a(k)$ converges and $\sum_{k=1}^\infty b(k)$ diverges then
     – $\sum_{k=1}^\infty [r\,a(k) + s\,b(k)]$ diverges as long as $s \ne 0$.

Exact Sums of Series

1. A geometric series converges if and only if $|r| < 1$ in which case the sum is given by the formula

$$\sum_{k=0}^\infty ar^k = \frac{a}{1-r}$$

2. Telescoping series can be summed by using partial fractions to 'telescope' the partial sums.

Tests that apply to any series

1. n th Term Test: An infinite series diverges if

$$\lim_{k\to\infty} a_k = \text{anything but } 0$$

   (a) Can be applied to any series
   (b) Can only inform that a series diverges – can never inform that a series converges

2. Absolute Value Series Test
   (a) If $\sum_k^\infty |a_k|$ converges then so does $\sum_k^\infty a_k$ and the latter is said to converge is absolutely.
      - Rearrangements of absolutely convergent series do not affect either the fact of convergence or the sum.
   (b) If $\sum_k^\infty |a_k|$ diverges and $\sum_k^\infty a_k$ converges then the latter is said to converge conditionally.
      - A conditionally convergent series may be rearranged to converge to any number or to diverge to either plus or minus infinity.

Tests that apply only to series with positive or non-negative terms

- p- series converge if and only if $p > 1$(but finding the actual sum is much harder)

$$\sum_{k=1}^{n} \frac{1}{k^p}.$$

- Integral Test

$$\sum_{k=1}^{\infty} f(k) \quad \text{and} \quad \int_{1}^{\infty} f(x)\, dx \quad \text{converge or diverge together}$$

  - Applies only for a positive, decreasing continuous function f

- Direct Comparison Test

  1. If $\sum_{k}^{\infty} c_k$ dominates $\sum_{k}^{\infty} a_k$ ($a_k \le c_k$ for all large k) and converges, then so does $\sum_{k}^{\infty} a_k$
  2. $\sum_{k}^{\infty} d_k$ is dominated by $\sum_{k}^{\infty} a_k$ ($d_k \le a_k$ for all large k) and diverges, then so does $\sum_{k}^{\infty} a_k$

- Limit Comparison Test

  1. If $\lim_{k\to\infty} \frac{a_k}{b_k} = L$
     (a) L finite and non-zero, then $\sum_{k}^{\infty} a_k$ and $\sum_{k}^{\infty} b_k$ converge or diverge together.
     (b) $L = 0$ and $\sum_{k}^{\infty} b_k$ converges then $\sum_{k}^{\infty} a_k$ converges
     (c) $L = \infty$ and $\sum_{k}^{\infty} b_k$ diverges then $\sum_{k}^{\infty} a_k$ diverges

- Ratio Test and Root Test

  1. If $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$ or $\lim_{k\to\infty} \sqrt[k]{a_k} = L$ where
     (a) $L < 1$ then $\sum_{k}^{\infty} a_k$ converges.
     (b) $L > 1$ then $\sum_{k}^{\infty} a_k$ diverges
     (c) $L = 1$ then no information

Tests that apply only to series whose terms alternate in sign

- Alternating Series Test

  1. Can only inform that an alternating series converges. Can never inform that an a series diverges.
  2. If $\sum_{k}^{\infty} a_k = \sum_{k}^{\infty} (-1)^k u_k$ with
     (a) $u_k > 0$
     (b) u_k a decreasing sequence
     (c) $\lim_{k\to\infty} u_k = 0$

     Then $\sum_{k}^{\infty} a_k = \sum_{k}^{\infty} (-1)^k u_k$ converges.
  3. Easy bound on error using an approximation:
     (a) If $\sum_{k=1}^{\infty} (-1)^k u_k$ converges to S, then $\left| S - \sum_{k=1}^{n} (-1)^k u_k \right| < u_{n+1}$

Power Series

- Any series in either of the forms below is a function with an interval for domain.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n \\ f(x) &= \sum_{n=0}^{\infty} a_n (x-a)^n \end{aligned}$$

- Any power series is a function and converges on one of the following sets (which is the domain of the function.)

  1. At only one point (the number a)

2. On a finite interval centered at the number $x = a$
3. On the entire real line.

- Apply either the Ratio or Root Tests **to the absolute value series** to detect the radius of convergence.
- Check the endpoints separately
- Power series can be differentiated and integrated term-by-term.
  1. After integrating or differentiating, the resulting series have the same Radius Of Convergence as the original series.
  2. After integrating or differentiating, the endpoints can behave differently than in the original.
- Power series can be multiplied by collecting on powers of x.
  1. The product of two power series converges only at those numbers that are in both intervals of convergence.

Taylor Series and Maclaurin Series

- Every infinitely differentiable function $f(x)$ gives rise to a power series centered at $x = a$

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(x-a)^n \quad \text{(Taylor Series)}$$
$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\,(x-0)^n \quad \text{(Maclaurin Series)}$$

- Any infinitely differentiable function $f(x)$ satisfies Taylor's formula

$$f(x) = P_n(x) + R_n(x)$$

where $P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)\,(x-a)^k$ and $R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)\,(x-a)^{n+1}$ for some c between a and x.

- Hence, a Taylor series $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)\,x^k$ equals its generating function $f(x)$ if and only if

$$\lim_{n\to\infty} R_n(x) = 0$$

- We can estimate the remainder $R_n(x)$ using

$$|R_n(x)| \le \frac{M}{(n+1)!}\,|x|^{n+1}$$

where M denotes the absolute maximum of $\left|f^{(n+1)}(t)\right|$ on the interval between a and x.

- A few known functions and the Taylor Series they equal include:

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k, \quad -1 < x < 1 \\
e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for all } x \\
\cos(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \quad \text{for all } x \\
\sin(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad \text{for all } x \\
(1+x)^m &= 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots
\end{aligned}$$

The last is the binomial series and converges:

1. (a) For all x if m is an **integer** that is positive.
   (b) For $-1 < x < 1$ if $m \le -1$
   (c) For $-1 \le x \le 1$ if $m > 0$ but m is **not** an integer.

(d) For $-1 < x \leq 1$ if $-1 < m < 0$.

- The Taylor series for many other functions can be computed 'easily' by noting that those functions are combinations of the above or the derivatives or integrals of the above.

1. **Example:**

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k, \quad -1 < x < 1 \\
\frac{1}{1+x^2} &= \sum_{k=0}^{\infty} \left(-x^2\right)^k, \quad -1 < x < 1 \\
&= \sum_{k=0}^{\infty} (-1)^k x^{2k}, \quad -1 < x < 1 \\
\text{so we have } \arctan(x) &= \int \frac{1}{1+x^2}\, dx = \int \sum_{k=0}^{\infty} (-1)^k x^{2k} \ dx \\
&= \sum_{k=0}^{\infty} \int (-1)^k x^{2k}\, dx = \sum_{k=0}^{\infty} (-1)^k \tfrac{x^{2k+1}}{2k+1} \quad -1 \leq x \leq 1 \\
\text{Convergence at } x &= 1 \text{ is checked separately.}
\end{aligned}$$