Mathematics 181 – B,C — Fall 2007

Answer Key — **Exam 1**

September 18, 2007

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Name

Technology used:______________________ Directions:

- Be sure to include in-line citations every time you use technology.
- Include a careful sketch of any graph obtained by technology in solving a problem.
- **Only write on one side of each page.**

Do any six (6) of the following problems

1. [8, 7 points] Do any **two** (2) of the following three options.

   (a) What are the antiderivatives of the following functions?

   i. $\sin(kx)$ has antiderivative $\int \sin(kx)\ dx = -\frac{1}{k}\cos(kx) + C$

   ii. $\sec^2(kx)$ has antiderivative $\int \sec^2(kx)\,dx = \frac{1}{k}\tan(kx) + C$

   iii. $\frac{1}{\sqrt{1-x^2}}$ has antiderivative $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$

   iv. $\frac{1}{1+x^2}$ has antiderivative $\int \frac{1}{1+x^2}dx = \arctan(x) + C$

   v. e^{kx} has antiderivative $\int e^{kx}dx = \frac{e^{kx}}{k} + C$

   (b) The accompanying table gives data for the velocity of a vintage sports car accelerating from 0 to 142 miles per hour in 36 seconds (10 thousandths of an hour). Use rectangles to estimate how far the car traveled during the 36 seconds it took to reach 142 mi/h. If you use your calculator on this problem be sure to also write out the formula for the estimate.

Time (h)	Velocity (mi/h)	Time (h)	Velocity (mi/h)
0.0	0	0.006	116
0.001	40	0.007	125
0.002	62	0.008	132
0.003	82	0.009	137
0.004	96	0.001	142
0.005	108		

   Using Right Endpoints: $(0.001)(40 + 62 + 82 + 96 + 108 + 116 + 125 + 132 + 137 + 142) = 1.04$

   Using Left Endpoints: $(0.001)(0 + 40 + 62 + 82 + 96 + 108 + 116 + 125 + 132 + 137) = 0.898$

   (c) Are any of the following equal? If so, which. You do not need to actually add up any of them. **No two are equal.**

   i. $\sum_{k=3}^{100}(k-1)^2 = 2^2 + 3^3 + \cdots + 99^2$

   ii. $\sum_{k=17}^{118}(k-19)^2 = (-2)^2 + (-1)^2 + 0^2 + 1^2 + \cdots + 99^2$

   iii. $\sum_{k=-96}^{1}(k-1)^2 = (-97)^2 + (-96)^2 + \cdots + (-1)^2 + 0^2$

   iv. $\sum_{k=-1}^{97}(k+3)^2 = 2^2 + 3^2 + 4^2 + \cdots + 99^2 + 100^2$

2. [15 points] Use the Riemann Sum process to:

(a) Find a formula for the upper sum for the function $f(x) = 1 + x^3$ over the interval $[0,2]$ obtained by partitioning $[0,2]$ into n equal subintervals. [Useful fact: $\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$]

Partitioning $[0,2]$ into n equal subintevals give $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

Selecting the Right endpoint of each subinterval gives $c_1 = \frac{2}{n}$, $c_2 = 2 \cdot \frac{2}{n}, \cdots, c_k = k \cdot \frac{2}{n}, \cdots, c_n = n \cdot \frac{2}{n}$

So $\sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} f\left(k \cdot \frac{2}{n}\right)\frac{2}{n} = \frac{2}{n}\sum_{k=1}^{n}\left[1 + \left(k \cdot \frac{2}{n}\right)^3\right] = \frac{2}{n}\sum_{k=1}^{n}\left[1 + \frac{8k^3}{n^3}\right]$

$= \frac{2}{n}\left[\sum_{k=1}^{n} 1 + \frac{8}{n^3}\sum_{k=1}^{n} k^3\right] = \frac{2}{n}\sum_{k=1}^{n} 1 + \frac{16}{n^4}\sum_{k=1}^{n} k^3 = \frac{2}{n}(n) + \frac{16}{n^4}\left(\frac{1}{4}n^2(n+1)^2\right)$

$= 2 + 4\left(\frac{n+1}{n}\right)\left(\frac{n+1}{n}\right) = 2 + 4\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n}\right)$

Take the limit of these sums as $n \to \infty$ to calculate the area under the curve over $[0,2]$.

$\lim_{n\to\infty} 2 + 4\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n}\right) = 2 + 4(1)(1) = 6.$

3. [15 points] Use the fact that $f(x) = 1 + x^3$ is monotone increasing over the interval $[0,2]$ to find an error bound for the estimate in part a. of the previous problem. Include any pertinent figures and write your answer as a function of n (the number of subintervals).

   Since the function is monotone increasing we can draw a figure where the errors accumulating in each subinterval of a Riemann sum are all completely contained in a rectangle of height $f(2) - f(0)$ and width $\Delta x = \frac{2}{n}$. Thus the total error must be less than the area of that rectangle: Error $\leq (f(2) - f(0))\frac{2}{n} = (9-1)\frac{2}{n} = \frac{16}{n}$.

4. [15 points] Example 1 of Section 5.3 in the textbook explains why the function below is not integrable on the interval $[0,1]$.

   (a) Explain why it is true that for any partition P of the interval $[0,1]$ it is possible to select points c_k in two different ways: one where the Riemann sum adds up to 1 and another where the Riemann sum adds up to 0.

   Every interval, no matter how small, contains both rational and irrational numbers. Hence given any partition P of the interval $[0,1]$ we can select a rational point c_k in the kth subinterval and we can also select an irrational point d_k in that subinterval. Then, since $f(c_k) = 1$ and $f(d_k) = 0$ we can see that the Riemann sums satisfy:

$$\begin{aligned}\sum_{k=1}^{n} f(c_k)\Delta x_k &= \sum_{k=1}^{n}(1)\Delta x_k = \sum_{k=1}^{n}\Delta x_k = b - a = 1\\ \sum_{k=1}^{n} f(d_k)\Delta x_k &= \sum_{k=1}^{n}(0)\Delta x_k = 0\end{aligned}$$

   (b) Explain why this is enough to show that the function is **not integrable.**

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational}\end{cases}$$

   Since the result of computing a Riemann sum **depends on how you select points in each subinterval**, the definition on page 333 is not satisfied and so this function f cannot be integrated.

5. [15 points] Do both of the following.

(a) Express as a definite integral where P is a partition of $[-3,-1]$.

$$\lim_{\|P\|\to 0}\sum_{k=1}^{n}\left(\tan^2(7c_k)\,e^{-c_k}\right)\Delta x_k$$

This limit, if it exists, represents the definite integral $\int_{-3}^{-1}\tan^2(7x)\,e^{-x}\,dx$

(b) Does this limit exist? Why? If you think it exists, do not bother to compute it.
The function $f(x)=\tan^2(7x)\,e^{-x}$ does not have the entire interval $[-3,-1]$ in its domain (there are vertical asymptotes at various points ($x=-\frac{\pi}{2}$ is one). Thus the limit does not exist.

6. [15 points] The Domination property of Table 5.3 in Section 5.3 of the text applies to integrable functions and reads

$$f(x)\ge g(x)\ \text{ on }\ [a,b]\ \text{ implies }\ \int_a^b f(x)\,dx\ge\int_a^b g(x)\,dx$$

(a) Use the full Riemann Sum process to explain why this is true.
Let P be any partition of the interval $[a,b]$ and make any selection of points c_k in the kth interval.
Then $f(c_k)\ge g(c_k)$ and since Δx_k is positive we have $f(c_k)\Delta x_k\ge g(c_k)\Delta x_k$
Thus $\sum_{k=1}^{n}f(c_k)\Delta x_k\ge\sum_{k=1}^{n}g(c_k)\Delta x_k$
Since this is true for every partition and every selection of points, then the following limits satisfy the same inequality
$\lim_{\|P\|\to 0}\sum_{k=1}^{n}f(c_k)\Delta x_k\ge\lim_{\|P\|\to 0}\sum_{k=1}^{n}f(c_k)\Delta x_k$
That is, $\int_a^b f(x)\,dx\ge\int_a^b g(x)\,dx$

7. [Math 181B] [15 points] Compute the following function values by using a well-known area formula.

(a) $F(2)$ where $F(x)=\int_0^x\sqrt{4-x^2}\,dx$
$y=\sqrt{4-x^2}$ is the graph of the top half of the circle of radius 2 centered at the origin.
$F(2)$ represents the area bounded above by this half circle, below by the x-axis and between 0 and 2.
This is 1/4 of the circle so $F(2)=\frac{1}{4}\pi(2)^2=\pi$

(b) $F(-2)$ where $F(x)=\int_0^x\sqrt{4-x^2}\,dx$
$F(-2)=\int_0^{-2}\sqrt{4-x^2}dx=-\int_{-2}^{0}\sqrt{4-x^2}dx$
The integral (without the minus sign) represents the area bounded by the circle and the x-axis between -2 and 0 which is also π
Thus $F(-2)=-\pi$

(c) Find $\frac{dy}{dx}$ for the function

$$y=\int_{\tan(x)}^{0}\frac{dt}{1+t^2}$$

$y=\int_{\tan(x)}^{0}\frac{dt}{1+t^2}=-\int_0^{\tan(x)}\frac{1}{1+t^2}dt$ so the derivative $\frac{dy}{dx}=\frac{-1}{1+\tan^2(x)}\cdot\frac{d}{dx}[\tan(x)]=\frac{-1}{1+\tan^2(x)}\cdot\sec^2(x)=\frac{-1}{\sec^2(x)}\cdot\sec^2(x)=-1$

7. [Math 181C] [15 points] The following denotes the area of a region in the plane. Carefully describe that region.

$$\lim_{\|P\|\to 0}\sum_{k=1}^{n}\left[9\left(2+\frac{5k}{n}\right)^5-\left(2+\frac{5k}{n}\right)^2+15\right]\frac{5}{n}$$

This limit represents a definite integral. The $\frac{5}{n}$ tells us that $\Delta x = \frac{5}{n}$ so we know the length of the original interval is 5.

Thus the definite integral equal to the given limit can be written either as $\int_2^7 \left(9x^5 - x^2 + 15\right)\, dx$ or $\int_0^5 \left(9\left(2+x\right)^5 - \left(2+x\right)^2 + 15\right)\, dx$.

This tells us that the region being described is the region between the graph of $y = 9x^5 - x^2 + 15$ and the x-axis between $x = 2$ and $x = 7$

Or, if you prefer, it is also the region between the graph of $y = 9\left(2+x\right)^5 - \left(2+x\right)^2 + 15$ and the x-axis between $x = 0$ and $x = 5$.