Mathematics 181 – D **Fall 2006**

Exam 4

November 14, 2006

____
Name

Technology used:____ Directions:

- Be sure to include in-line citations every time you use technology.
- Include a careful sketch of any graph obtained by technology in solving a problem.
- **Only write on one side of each page.**
- **When given a choice, specify which problem(s) you wish graded.**

Do any six (6) of the following problems

1. (15 points) Use the error bound formula for the Trapezoid Rule, $|E_T| \leq \frac{M(b-a)^3}{12n^2}$ to estimate the minimum number of subintervals needed to approximate the integral $\int_0^2 \sqrt{x+1}\,dx$ with an error of magnitude less than 10^{-4}.

   (a) $f(x) = (x+1)^{1/2}$, so $f'(x) = \frac{1}{2}(x+1)^{-1/2}$, $f'' = -\frac{1}{4}(x+1)^{-3/2}$ and $|f''(x)| = \frac{1}{4}\frac{1}{|x+1|^{3/2}}$. This last function has a decreasing graph on the interval $[0,2]$ so $|f''(x)| \leq |f''(0)| = \frac{1}{4}$ which we use for M.
   We want n so that $\frac{\frac{1}{4}(2-0)^3}{12n^2} \leq 10^{-4}$ which is equivalent to $\frac{\frac{1}{4}(2-0)^3}{12}10^4 \leq n^2$ and taking square roots we have $40.824 \leq n$ so we use $n = 41$.

2. (15 points) Does the following integral coverge or diverge? Show all work.

$$\int_{-\infty}^{\infty} \frac{x\ dx}{(x^2+9)^{3/2}}$$

   (a) Using the substitution $u = x^2 + 9$ we see that $\int \frac{x\,dx}{(x^2+9)^{3/2}} = \frac{-1}{(x^2+9)^{1/2}} + C$
   $\int_{-\infty}^{\infty} \frac{x\ dx}{(x^2+9)^{3/2}} = \int_{-\infty}^{0} \frac{x\ dx}{(x^2+9)^{3/2}} + \int_0^{\infty} \frac{x\ dx}{(x^2+9)^{3/2}} = \lim_{a\to-\infty} \int_a^0 \frac{x\ dx}{(x^2+9)^{3/2}} + \lim_{b\to\infty} \int_0^b \frac{x\ dx}{(x^2+9)^{3/2}} =$
   $\lim_{a\to-\infty} \left[\frac{-1}{(x^2+9)^{1/2}}\right]_a^0 + \lim_{b\to\infty} \left[\frac{-1}{(x^2+9)^{1/2}}\right]_0^b$
   $= \lim_{a\to-\infty} \left[\frac{-1}{(0^2+9)^{1/2}} - \frac{-1}{(a^2+9)^{1/2}}\right] + \lim_{b\to\infty} \left[\frac{-1}{(b^2+9)^{1/2}} - \frac{-1}{(0^2+9)^{1/2}}\right] = \left[\frac{-1}{3} - 0\right] + \left[0 - \frac{-1}{3}\right]$. And since both improper integrals converge the original integral converges to 0.

3. Below are six infinite series,

   (a) (2 points each) For **five** (5) of the six, state a reasonable test for checking for convergence or divergence and include a short sentence as to why that test is reasonable.

   (b) (10 points each) Choose **three** (3) of the series and determine if they converge or diverge. Show all details.

      i. **Ratio or Root test or write as a sum of geometric series :** $\sum_{n=1}^{\infty} \frac{6^n+1}{7^n}$

A. $\sum_{n=1}^{\infty} \frac{6^n+1}{7^n} = \sum_{n=1}^{\infty} \left(\frac{6}{7}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{7}\right)^n$ both of which converge as geometric series.

ii. **Ratio or Root Test or nth term test:** $\sum_{n=1}^{\infty} \frac{7^n}{6^n+1}$

A. $\lim_{n\to\infty} \frac{7^n}{6^n+1} = \lim_{n\to\infty} \frac{(7/6)^n}{1+\left(\frac{1}{6}\right)^n} = \infty$ so the series diverges by the nth term test.

iii. **Direct comparison to the harmonic series or integral test:** $\sum_{n=1}^{\infty} \frac{[\ln(n)]^3}{n}$

A. $\frac{1}{n} \leq \frac{[\ln(n)]^3}{n}$ for all n larger than 1 so by direct comparison with the divergent harmonic series, the series $\sum_{n=1}^{\infty} \frac{[\ln(n)]^3}{n}$ diverges.

iv. **LImit comparison to $\sum \frac{1}{n^3}$:** $\sum_{n=2}^{\infty} \frac{6n^3-10n^2+1000n}{n^6-1}$

A. $\lim_{n\to\infty} \frac{6n^3-10n^2+1000n}{n^6-1} \cdot \frac{n^3}{1} = \lim_{n\to\infty} \frac{6-\frac{10}{n}+\frac{1000}{n^2}}{1-\frac{1}{n^6}} = 6$ so by limit comparison to the converging series $\sum \frac{1}{n^3}$ the series $\sum_{n=2}^{\infty} \frac{6n^3-10n^2+1000n}{n^6-1}$ also converges.

v. **Root test is reasonable but won't work. nth term test works:** $\sum_{n=1}^{\infty} \left(\frac{n-4}{n}\right)^n$

A. $\lim_{n\to\infty} \left(\frac{n-4}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{-4}{n}\right)^n = e^{-4}$ (See section 8.2). So this series diverges by the nth term test.

vi. **Ration Test:** $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$ [Be careful with the factorials.]

A. Using the ratio test, $\lim_{n\to\infty} \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{(n+1)!} = \lim_{n\to\infty} \frac{(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty} \lim_{n\to\infty} \frac{1}{2(2n+1)} = 0$. So by the ratio test, this series converges.

4. (15 points) Determine if the following series converges or diverges. If it converges, determine if the convergence is absolute or conditional.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n)}{n^3}$

(a) The absolute value series is $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^3}$ and $\frac{|\sin(n)|}{n^3} \leq \frac{1}{n^3}$ for all n so $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^3}$ converges by direct comparison to $\sum_{n=1}^{\infty} \frac{1}{n^3}$ and then $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n)}{n^3}$ converges absolutely by the Absolute Convergence Test.

5. (10 points) Prove the following theorem.

If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ also converges.

(a) Since $\frac{a_n}{n} \leq a_n$ for all positive integers n and $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges by direct comparison with $\sum_{n=1}^{\infty} a_n$.