Mathematics 433-A **Fall 2006**

Exam 2

November 14, 2006

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Name

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Directions: Only write on one side of each page.

Do any six (6) of the following

1. Let $K \subset H \subset G$ be subgroups of a **finite** group G. Prove the formula

$$[G : K] = [G : H] [H : K] .$$

   (a) The basic formla is that $|G| = [G : H] |H|$ for every group G and every subgroup H of G. Thus

$$[G : K] = \frac{|G|}{|K|} = \frac{|G|}{|H|} \frac{|H|}{|K|} = [G : H] [H : K] .$$

2. Do **both** of the following:

   (a) If S is a set and G is a group acting on S, prove that the relation

$$s \sim s' \text{ if } s' = gs \text{ for some } g \in G$$

   is an equivalence relation.

   i. Reflexive: $es = s$, symmetric: $gs_1 = s_2$ implies $s_1 = g^{-1} s_2$, transitive: $gs_1 = s_2$ and $hs_2 = s_3$ implies $(hg)\, s_1 = h\,(gs_1) = s_3$

   (b) Let $\phi : G \to G\,'$ be a homomorphism, and let S be a set on which $G\,'$ acts. Use ϕ to define, with proof, a group action of G on S.

   i. Given the group action $g's$ for the group G' define a group action for the group G by $gs = \phi(g)\, s = g's$ where $\phi(g) = g'$.Then $es = \phi(e)\, s = e's = s$ and $g_2\,(g_1 s) = g_2\,(g_1' s) = \phi(g_2)\,(g_1' s) = g_2'\,(g_1' s) = (\phi(g_1)\,\phi(g_2))\, s = \phi(g_2 g_1)\, s = (g_2 g_1)'\, s = (g_2 g_1)\, s$.

3. Do one of the following

   (a) Let G be a group containing normal subgroups of orders 3 and 5, respectively. Prove G contains an element of order 15.

   i. Let H have order 3 and K have order 5. Normality of H and K in G tells us that $hkh^{-1}k^{-1} = (hkh^{-1})\, k^{-1} = h\,(kh^{-1}k^{-1})$ is an element in $H \cap K$. But $H \cap K \leq H$ and $H \cap K \leq K$ so $|H \cap K|$ divides both 3 and 5 and so there is only one element, e, in $H \cap K$. Thus, $hkh^{-1}k^{-1} = e$ so $hk = kh$ for every $h \in H$ and $k \in K$. Now let a be the generator of the cyclic group H and k the generator of the cyclic group K. The element ab of G satisfies $(ab)^n = a^n b^n$ and the smallest positive number n for which this yields e must have both 3 and 5 as divisors. Thus $n = 15$ is the order of the element ab in G.

   (b) Let H, K be subgroups of a group G. Show the set of products $HK = \{hk : h \in H,\ k \in K\}$ is a subgroup if and only if $HK = KH$.

i. " $\rightarrow$" If $HK \leq G$

A. $kh = \left(h^{-1}k^{-1}\right)^{-1} \in HK$ by closure of subgroups and $KH \subset HK$.

B. KH is closed under inverses since $(kh)^{-1} = h'k' \in HK$ so $hk = \left(k^{-1}h^{-1}\right)^{-1} \in KH$ and $KH = HK$

ii. " $\leftarrow$" If $HK = KH$ then $(h_1k_1)(h_2k_2)^{-1} = h_1\left(k_1h_2^{-1}\right)k_2^{-1} = h_1(h_3k_3)k_2 \in HK$ so HK is a subgroup by the 1 step theorem.

4. Do one of the following:

When we classified the group M of rigid motions of the plane we claimed the following six relations were all true and proved a few of them.

(a) Add to our certainty by **algebraically** proving either part iv. or part v.

i. $t_a t_b = t_{a+b}$

ii. $\rho_\theta \rho_\eta = \rho_{\theta+\eta}$

iii. $rr = i$

iv. $\rho_\theta t_a = t_{a'}\rho_\theta$, where $a' = \rho_\theta(a)$

v. $rt_a = t_{a'}r$, where $a' = r(a)$

A. Write out $rt_a(x)$ and $t_{a'}r(x)$ using the functions below to see that they are equal for all x.

vi. $r\rho_\theta = \rho_{-\theta}r$.

(b) Use the above relations to show that if m is an orientation reversing motion of the plane then m^2 is a translation.

i.

$$\begin{aligned} m^2 &= (t_a\rho r)(t_a\rho r) = t_a\rho t_{a'} r\rho r = \\ &= t_a\rho t_{a'}\rho^{-1}rr = t_a\rho t_{a'}\rho^{-1} \\ &= t_a t_{a''}\rho\rho^{-1} \\ &= t_{a+a''} \end{aligned}$$

(c) Compute the glide vector of the glide $t_{\vec{a}}\rho_\theta r$ in terms of $\vec{a}$ and θ.

5. Do one of the following:

(a) Let G be a group and $Aut(G)$ the group of automorphisms of G. Prove or disprove: The set of inner automorphisms $Inn(G) = \{\phi \in Aut(G) : \phi(g) = xgx^{-1}$ for some $x \in G\}$ is a normal subgroup of $Aut(G)$.

i. $\phi, \psi \in Inn(G)$ where $\phi(g) = xgx^{-1}$ and $\psi(g) = ygy^{-1}$, then $\phi \circ \psi^{-1}(g) = \phi(y^{-1}gy) = xy^{-1}gyx^{-1} = (xy^{-1})g(xy^{-1})^{-1} \in Inn(G)$ so $Inn(G)$ is a subgroup by the one step test.

ii. Let f be an automorphism and $\phi \in Inn(G)$ as above then $f\phi f^{-1}(g) = f(xf^{-1}(g)x^{-1}) = f(x)f(f^{-1}(g))f(x^{-1}) = (f(x))g(f(x))^{-1} \in Inn(G)$

(b) Let S be a set on which a group G operates. Let $H = \{g \in G : gs = s$ for all $s \in S\}$. Prove H is a normal subgroup of G.

i. Let $x \in G$ and $h \in H$ then $(xhx^{-1})s = x(h(x^{-1}s)) = x(x^{-1}(s)) = es = s$ so H is normal in G.

6. The following patterns represent small portions of two tilings of the infinite plane. Circle one of the following patterns and let G be the group of symmetries of that tiling. Determine the point group of G.

Figure 1: 1 2

(a) See attached image file for the patterns

(b) #1: If the pattern repeats to cover the plane then the smallest angle of rotation is π and there are horizontal reflections so the point group is D_2 If the pattern is restricted to an infinte horizontal strip then the smallest angle of rotation is 2π and the point group is D_1

(c) #2: Similarly, for a plane-filling pattern the point group is D_2 and for an infinite strip it is again D_2

7. Let G be a group acting on the set S. Let s be a fixed element in S and t an element in the orbit of s, say $t = as$. Prove the stabilizer of t in G is a conjugate subgroup of the stabilizer of s in G. Specifically, show $G_t = aG_s a^{-1}$.

   (a) $g \in G_t$ iff $gt = t$ iff $g(as) = as$ iff $(a^{-1}ga)s = s$ iff $a^{-1}ga \in G_s$ iff $g = a(a^{-1}ga)a^{-1} \in aG_s a^{-1}$.

8. Determine the group of automorphisms $Aut(G)$ if $G = C_2 \times C_2$.

Useful Facts

- $$\rho_\theta\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
  $$r\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$