Mathematics 122 A,B Fall 2004 Quiz 7

Due November 12

______________________________ Study Group Members   ______________________________ Name

Only write on one side of each page.
I encourage you to work with others in the class on this quiz. As with all writing you should work out the details in a draft before writing a final solution. Be sure to follow the 5 basic guidelines listed in the course information sheet unless explicitly directed to do otherwise in the problem statement. You do not need to include every algebra or arithmetic step but you should include enough detail to allow a member of your target audience to reconstruct any missing steps. Be sure to include in-line citations, with page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. If you include graphs, they should be done carefully on graph paper. Finally, there is to be no collaboration in the writing of your solution even if you worked out the details with other people.
"*One is always a long way from solving a problem until one actually has the answer*". – Stephen Hawking

Problems

1. In the following you will explore some of the properties of the Gamma Function $\Gamma(x)$. One of the tools you will need is the fact (proven later in the semester) that if a is a positive constant then

$$\lim_{x\to\infty} \frac{x^a}{e^x} = 0.$$

Another tool you will need is the fact that the improper integral

$$\int_0^\infty e^{-x^2}\,dx$$

converges to the value $\sqrt{\pi}/2$. (This is proven in multivariable calculus.)

The Gamma function is defined as follows:

$$\begin{aligned} \Gamma &: (0,\infty) \longrightarrow R \\ \Gamma &: x \mapsto \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\ dt. \end{aligned}$$

For example, note that

$$\begin{aligned} \Gamma(2) &= \int_0^\infty t^{2-1}e^{-t}\ dt \\ &= \int_0^\infty te^{-t}\ dt. \end{aligned}$$

This is an improper integral that can be evaluated by using integration by parts.

(a) Directly evaluate the improper integral for $\Gamma(1)$.

(b) Use the substitution $t = u^2$ to evaluate the improper integral for $\Gamma(1/2)$.

(c) Use integration by parts to show that for any $x > 0$,

$$\Gamma(x+1) = x\Gamma(x)$$

(d) Without directly evaluating any improper integrals, compute $\Gamma(2)$, $\Gamma(3)$, $\Gamma(3/2)$, $\Gamma(5/2)$, and $\Gamma(-1/2)$.

(e) Explain why some people use "$\frac{1}{2}!$" when they refer to $\Gamma(3/2)$. What would these people say when they refer to $\Gamma\left(\frac{11}{6}\right)$?