Solution to Number 63 page 187 of Bradley,Smith

63. Find all points on the cardioid

$$\left(x^2+y^2\right)^{3/2} = \left(x^2+y^2\right)^{1/2} + x$$

where the tangent line is vertical.

0.0.1 Solution

We find the points where dy/dx is undefined by solving for dy/dx and determining where the denominator equals 0.
Taking the derivative of both sides of the equation with respect to x, we have

$$\begin{aligned}
\frac{d}{dx}\left[\left(x^2+y^2\right)^{3/2}\right] &= \frac{d}{dx}\left[\left(x^2+y^2\right)^{1/2}+x\right] \\
\frac{3}{2}\left(x^2+y^2\right)^{1/2}\left(2x+2y\frac{dy}{dx}\right) &= \frac{1}{2}\left(x^2+y^2\right)^{-1/2}\left(2x+2y\frac{dy}{dx}\right)+1 \\
\frac{3}{2}\left(x^2+y^2\right)^{1/2}(2x)+\frac{3}{2}\left(x^2+y^2\right)^{1/2}\left(2y\frac{dy}{dx}\right) &= \frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x)+\frac{1}{2}\left(x^2+y^2\right)^{-1/2}\left(2y\frac{dy}{dx}\right)+1
\end{aligned}$$

Simplifying, we have

$$3x\left(x^2+y^2\right)^{1/2}+3y\left(x^2+y^2\right)^{1/2}\frac{dy}{dx} = x\left(x^2+y^2\right)^{-1/2}+y\left(x^2+y^2\right)^{-1/2}\frac{dy}{dx}+1.$$

Collecting terms with dy/dx, we obtain

$$\begin{aligned}
3y\left(x^2+y^2\right)^{1/2}\frac{dy}{dx}-y\left(x^2+y^2\right)^{-1/2}\frac{dy}{dx} &= x\left(x^2+y^2\right)^{-1/2}+1-3x\left(x^2+y^2\right)^{1/2} \\
\left(3y\left(x^2+y^2\right)^{1/2}-y\left(x^2+y^2\right)^{-1/2}\right)\frac{dy}{dx} &= x\left(x^2+y^2\right)^{-1/2}+1-3x\left(x^2+y^2\right)^{1/2} \\
\frac{dy}{dx} &= \frac{x\left(x^2+y^2\right)^{-1/2}+1-3x\left(x^2+y^2\right)^{1/2}}{3y\left(x^2+y^2\right)^{1/2}-y\left(x^2+y^2\right)^{-1/2}}.
\end{aligned}$$

As indicated above, the points we are interested in correspond to those points (x, y) where the denominator of the last fraction above equals zero. That is, we solve

$$\begin{aligned}
3y\left(x^2+y^2\right)^{1/2}-y\left(x^2+y^2\right)^{-1/2} &= 0 \\
3y\left(x^2+y^2\right)^{1/2}-\frac{y}{y\left(x^2+y^2\right)^{1/2}} &= 0 \\
\frac{3y\left(x^2+y^2\right)-y}{y\left(x^2+y^2\right)^{1/2}} &= 0.
\end{aligned}$$

Since a fraction can equal zero only when its numerator is zero, we have

$$\begin{aligned}
3y\left(x^2+y^2\right)-y &= 0 \\
y\left(3\left(x^2+y^2\right)-1\right) &= 0
\end{aligned}$$

which is true whenever $y=0$ or $\left(3\left(x^2+y^2\right)-1\right)=0$.

Case 1. In the case where $y = 0$, the equation of the cardiod

$$\left(x^2+y^2\right)^{3/2} = \left(x^2+y^2\right)^{1/2} + x$$

simplifies to

$$\begin{aligned} \left(x^2+0\right)^{3/2} &= \left(x^2+0\right)^{1/2} + x \\ |x|^3 &= |x| + x. \end{aligned}$$

Now, if x is negative, this last equation is

$$\begin{aligned} (-x)^3 &= (-x) + x \\ -x^3 &= 0 \\ x &= 0. \end{aligned}$$

If x is non-negative, $|x|^3 = |x| + x$ becomes

$$\begin{aligned} x^3 &= 2x \\ x^3 - 2x &= 0 \\ x\left(x^2-2\right) &= 0 \\ x &= 0, \sqrt{2}, -\sqrt{2} \end{aligned}$$

Note that $-\sqrt{2}$ cannot be a solution because we are in the case when x is non-negative.
Thus $y = 0$ leads to the points $(0,0)$ and $\left(\sqrt{2},0\right)$ as places where the derivative $\frac{dy}{dx}$ fails to exist. To find the remaining places we still need to look at the remaining case.

Case 2. In this case, we have

$$\begin{aligned} 3\left(x^2+y^2\right) - 1 &= 0 \\ \left(x^2+y^2\right) &= \frac{1}{3}. \end{aligned}$$

Thus the original equation for the cardioid simplifies in the following fashion.

$$\begin{aligned} \left(x^2+y^2\right)^{3/2} &= \left(x^2+y^2\right)^{1/2} + x \\ \left(\frac{1}{3}\right)^{3/2} &= \left(\frac{1}{3}\right)^{1/2} + x \\ x &= \left(\frac{1}{3}\right)^{3/2} - \left(\frac{1}{3}\right)^{1/2} \end{aligned}$$

So, for this value of x we have $(x^2+y^2) = \frac{1}{3}$ which reduces to

$$\begin{aligned} \left(\left(\frac{1}{3}\right)^{3/2} - \left(\frac{1}{3}\right)^{1/2}\right)^2 + y^2 &= \frac{1}{3} \\ y^2 &= \frac{1}{3} - \left(\left(\frac{1}{3}\right)^{3/2} - \left(\frac{1}{3}\right)^{1/2}\right)^2 \\ &= \frac{1}{3} - \left(\frac{1}{27} - 2\left(\frac{1}{9}\right) + \left(\frac{1}{3}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3} - \left(\frac{1-6+9}{27}\right) \\
&= \frac{1}{3} - \frac{4}{27} \\
&= \frac{5}{27}.
\end{aligned}
$$

This means that there are two more points where dy/dx is not defined. Those points are

$$\left(\left(\frac{1}{3}\right)^{3/2} - \left(\frac{1}{3}\right)^{1/2}, \sqrt{\frac{5}{27}}\right) \quad \text{and} \quad \left(\left(\frac{1}{3}\right)^{3/2} - \left(\frac{1}{3}\right)^{1/2}, -\sqrt{\frac{5}{27}}\right).$$