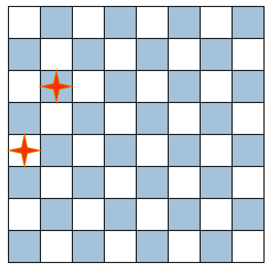
CONSTRAINT SATISFACTION

Today

- Reading
  - AIMA Read Chapter 6.1-6.3, Skim 6.4-6.5

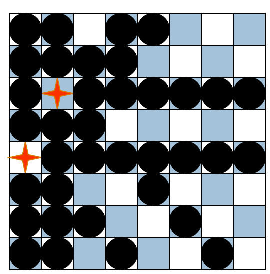
- Goals
  - Constraint satisfaction problems (CSPs)
  - Types of CSPs
  - Inference
  - (Search + Inference)

8-queens problem



How would you go about deciding where to put a third queen on the board in column 3?

8-queens problem



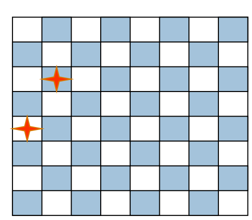
How would you go about deciding where to put a third queen on the board in column 3?

Constraint satisfaction problems

- Set of variables $\{X_1, X_2, \ldots, X_n\}$
- Each variable X_i has a domain D_i of possible values
- Set of constraints $\{C_1, C_2, \ldots, C_p\}$
  - Each constraint C_k involves a subset of variables and specifies the allowable combinations of values to these variables
- A state is an assignment of values to some or all of the variables
  - If the assignment doesn't violate any constraints we say it is consistent or legal
- The goal test is checking for a consistent and complete assignment

Example: 8-queens problems

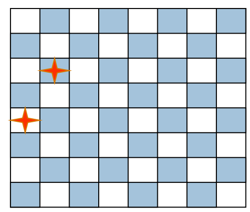
- Variable?
- Domain?
- Constraints?



Example: 8-queens problems

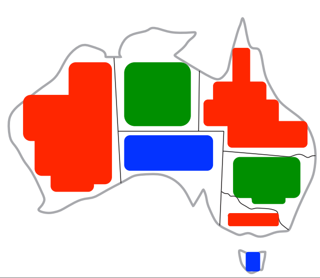
- Variables: one for each queen $\{X_1, \ldots, X_8\}$
- Domain: indicates row D = {1,2,…,8}
- Constraints:

$$X_i = k \Longrightarrow X_j \neq k \quad \forall i \neq j$$
$$X_i = k_i, X_j = k_j \Longrightarrow |i-j| \neq |k_i - k_j|$$



Example: Map coloring

- Variables: {WA, NT, SA, Q, NSW, V, T}
- Domains: {red, blue, green}
- Constraints: adjacent regions have different colors
  - Implicit: WA ≠ NT, WA ≠ SA, SA ≠ NT, NT ≠ Q,…
  - Explicit: (WA,NT) ∈ {(red,green), (red,blue),…}

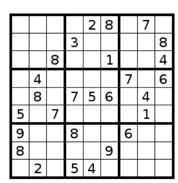


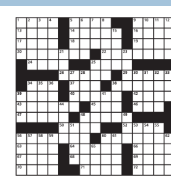
Example: Task scheduling

- Variables: {AxleF, AxleB, WheelRF, WheelLF,…,Inspect}
- Domains: Time task starts D = [0, 1, 2,…, ∞)
- Constraints:
  - Axle must be done before the wheel
    - AxleF + 10 < WheelLF
    - AxleF + 10 < WheelRF
  - The front axle and the back axle cannot be done at the same time
    - (AxleF + 10 < AxleB) OR (AxleB + 10 < AxleF)
  - Everything must be done within 30 minutes
    - Change domains to have upper bound 30 min.

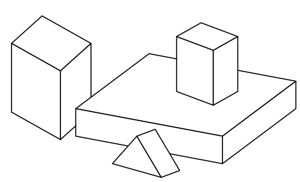
More examples

- More toy examples
  - sudoku, cryptarithmetic





- Real-world applications
  - Interpreting lines in 3D
  - Assignment problems, e.g. who teaches what class?
  - Timetable problems, e.g. which class offered when? where?
  - Transportation scheduling
  - Factory scheduling
  - Circuit layout

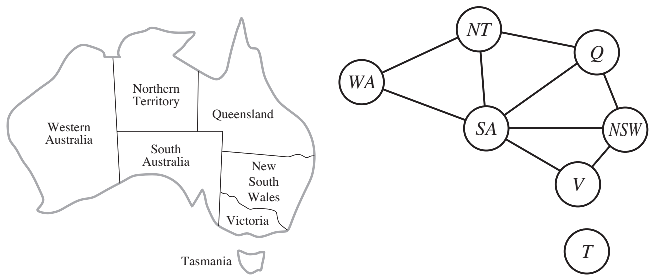


Types of CSPs - constraints

- Unary constraints involve a single variable
  - e.g. SA ≠ green
- Binary constraints involve pairs of variables
  - SA ≠ NSW
  - A binary CSP can be illustrated using a constraint graph
- Higher-order constraints
  - e.g. A, B, and C cannot be in the same grouping
  - e.g. AllDiff (all variables must be assigned different values)
- Preference constraints
  - costs on individual variable assignments
  - constraint optimization problem

Constraint Graph

- Useful for binary constraint CSPs where each constraint relates (at most) two variables
- Nodes correspond to variables
- Edges (arcs) link two variables that participate in a constraint
- Use graph to speed up search



Solving CSPs: Constraint Propagation

- Use the constraints to reduce the number of legal values for a variable
- Possible to find a solution without searching
  - Node consistency
    - A node is node-consistent if all values in its domain satisfy the unary constraints
  - Arc consistency
    - A node X_i is arc-consistent w.r.t. node X_j if for every value in D_i there exists a value in D_j that satisfies the binary constraint
    - Algorithm AC-3
  - Other types of consistency (path consistency, k-consistency, global constraints)

AC-3 algorithm for Arc consistency

```
function AC-3(csp) returns false if inconsistency found, true otherwise
    queue ⟵ all arcs in csp
    while queue not empty
        (Xi,Xj) ⟵ REMOVE-FIRST(queue)
        if REMOVE-INCONSISTENT-VALUES(Xi, Xj)
            if size Di == 0 return false
            for each arc (Xk ,Xi)
                add (Xk, Xi) to queue
    return true
```

```
function REMOVE-INCONSISTENT-VALUES(Xi, Xj)
    revised ⟵ false
    for each x in Di
        if ∄ y in Dj s.t. (x,y) satisfies constraints
            delete x from Di
            revised ⟵ true
    return revised
```

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c constraints (arcs)
d domain size

$O(cd)$

Total $O(cd^3)$

```
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    revised ⟵ false
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        if ∄ y in Dj s.t. (x,y) satisfies constraints
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```
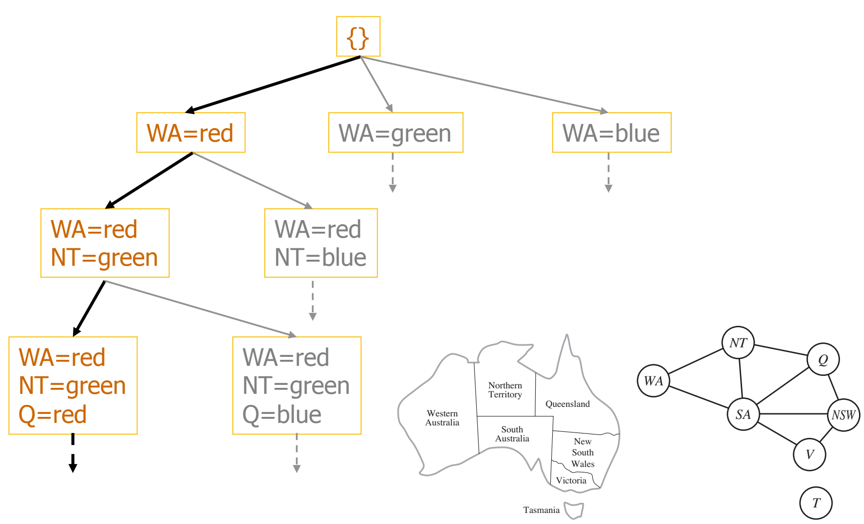

$O(d^2)$

Search



Backtracking Search

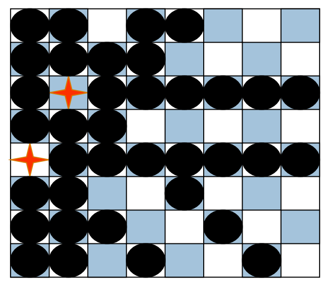
```
function CSP-BACKTRACKING(assignment) returns a solution or failure
    if assignment complete return assignment
    X ⟵ [select unassigned variable]
    D ⟵ [select an ordering for the domain of X]
    for each value in D
        if value is consistent with assignment
            add (X = value) to assignment
            (ADD INFERENCE HERE)
            result ⟵ CSP-BACKTRACKING(assignment)
            if result ≠ failure return result
        remove (X = value) from assignment
    return failure
```

Improving Backtracking search

- Idea 1: Intelligent ordering
  - Which variable X should be assigned a value next?
  - In which order should its domain D be sorted?
- Idea 2: Incorporating inference
  - Forward checking
  - AC-3
- Idea 3: Exploiting structure
  - Can we exploit the problem structure?

Idea 1: Intelligent Ordering

- Which variable should we choose?



Idea 1: Intelligent Ordering

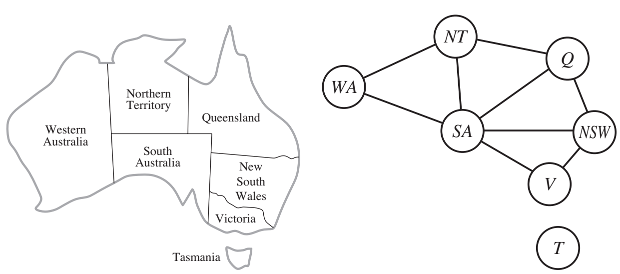
- Variable ordering
  - Minimum-remaining values heuristic - Choose the variable with the fewest "legal" moves remaining
  - Degree heuristic - Choose variable involved in the largest number of constraints with remaining unassigned variables
- Value ordering
  - Least-constraining value heuristic - Choose the value that rules out the fewest choices for the neighboring variables

Idea 2: Incorporating Inference

- Forward checking
  - After an assignment X = x, ensure all arcs of the form (Y,X) are arc consistent
- Run AC-3 algorithm
  - Ensure all arcs are arc consistent
- Run path-consistency or k-consistency algorithm

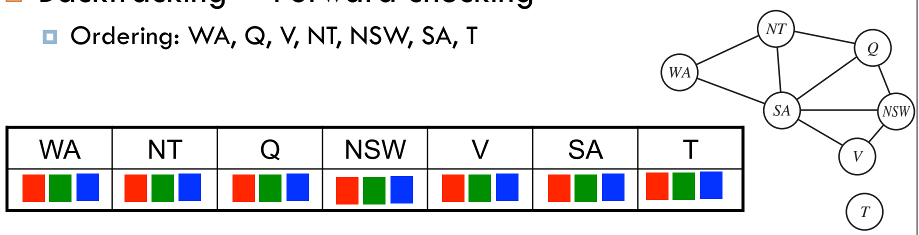
Example

- Run Backtracking on graph coloring
  - Use fixed ordering of variables
  - Use forward checking for inference



Limitations of Forward Checking

- Backtracking + Forward checking
  - Ordering: WA, Q, V, NT, NSW, SA, T



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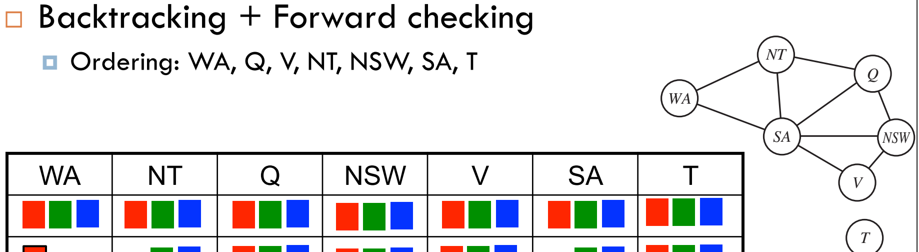

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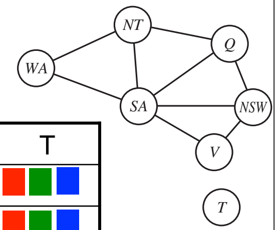

WA	NT	Q	NSW	V	SA	T
R G B	R G B	R G B	R G B	R G B	R G B	R G B
R	G B	R G B	R G B	R G B	G B	R G B
R	B	G	R B	R G B	B	R G B

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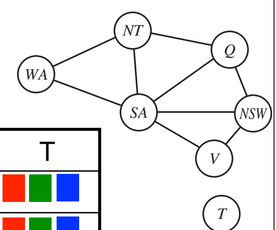

WA	NT	Q	NSW	V	SA	T
R G B	R G B	R G B	R G B	R G B	R G B	R G B
R	G B	R G B	R G B	R G B	G B	R G B
R	B	G	R B	R G B	B	R G B
R	B	G	R	B	✗	R G B

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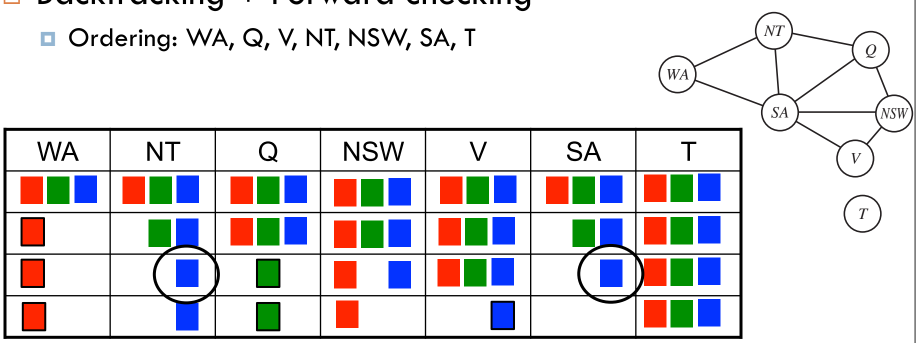
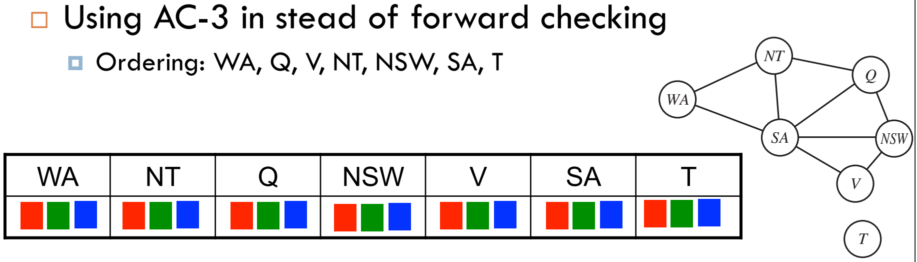


Could have detected earlier that things were going wrong!

Limitations of Forward Checking

- Using AC-3 in stead of forward checking
  - Ordering: WA, Q, V, NT, NSW, SA, T



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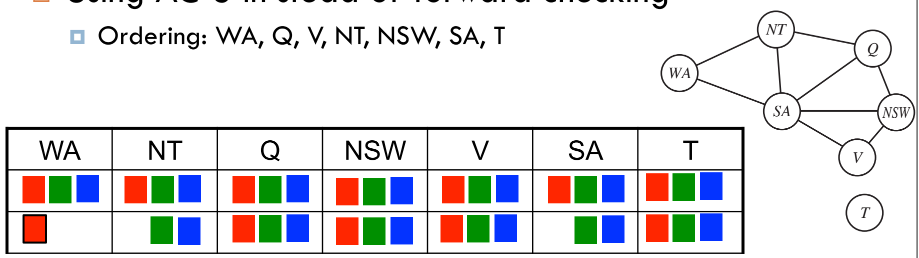

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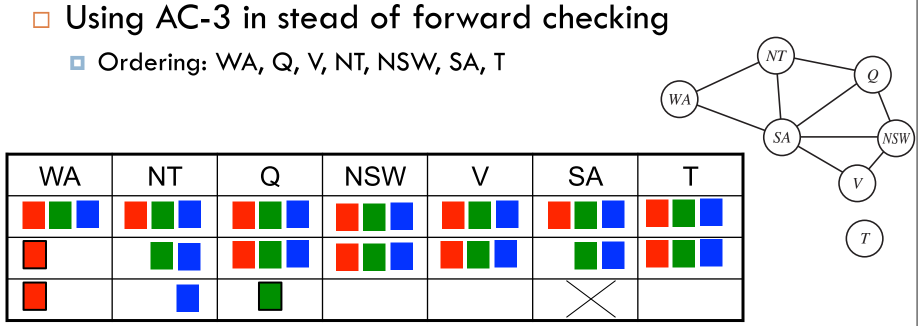
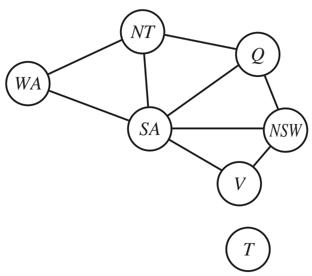


Idea 3: Exploit Structure

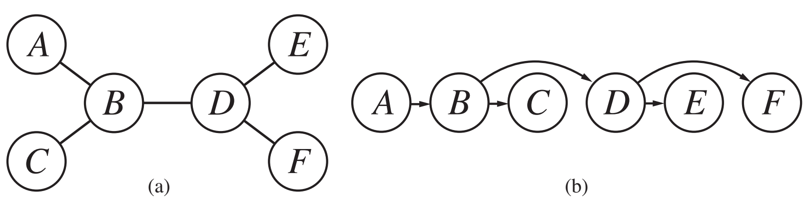
- Independent subproblems
  - Find connected components of the constraint graph
  - If we split n variables into sub-problems of c variables each then: $O(d^n) \longrightarrow O(d^c\, n/c)$



Tasmania is independent of the mainland

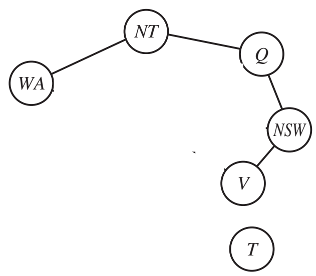
Idea 3: Exploit Structure

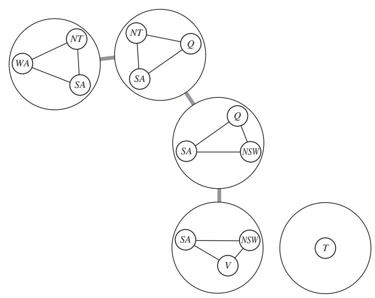
- Tree structured constraint graphs
  - Can solve in linear time using AC-3



Idea 3: Exploit Structure

- Reduction to a tree structured graph
  - Cycle cutset – a subset of the variables whose removal creates a tree.
  - Tree decomposition – Divide graph into subproblems, solve independently merge the solutions





CSP Summary

- Constraint Satisfaction Problems (CSPs)
- Solving CSPs using inference
- Solving CSPs using search
  - Backtracking algorithm = DSF + fixed ordering + constraints checking
  - General (not problem-specific) heuristics
- Improving Backtracking
  - Intelligent ordering
  - Incorporating inference
  - Exploiting structure