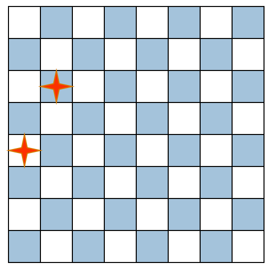
CONSTRAINT SATISFACTION

Today

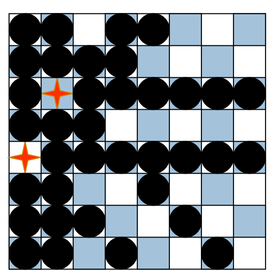
- Reading
  - AIMA Chapter 6
- Goals
  - Constraint satisfaction problems (CSPs)
  - Types of CSPs
  - Inference
  - Search + Inference

8-queens problem



How would you go about deciding where to put a third queen on the board in column 3?

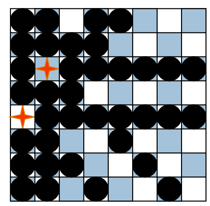
8-queens problem



How would you go about deciding where to put a third queen on the board in column 3?

8-queens problem

- This problem includes a set of constraints
- As a result, we need more than just a successor function and goal test
- We need a way to propagate the constraints imposed by one queen to the others and a way to detect early failure
  - Explicitly represent constraints
  - Algorithm to manipulate constraints

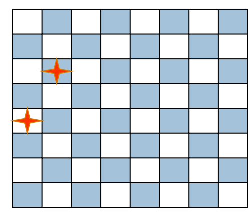


Constraint satisfaction problems

- Set of variables $\{X_1, X_2, \ldots, X_n\}$
- Each variable X_i has a domain D_i of possible values
- Set of constraints $\{C_1, C_2, \ldots, C_p\}$
  - Each constraint C_k involves a subset of variables and specifies the allowable combinations of values to these variables
- A state is an assignment of values to some or all of the variables
  - If the assignment doesn't violate any constraints we say it is consistent or legal
- The goal test is checking for a consistent and complete assignment

Example: 8-queens problems

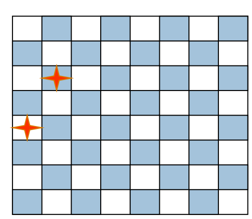
- Variable?
- Domain?
- Constraints?



Example: 8-queens problems

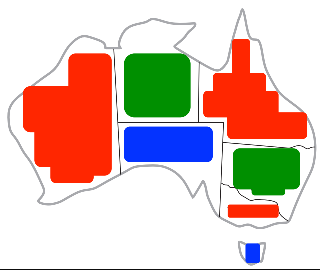
- Variables: one for each queen $\{X_1, \ldots, X_8\}$
- Domain: indicates row D = {1,2,…,8}
- Constraints:

$$X_i = k \Longrightarrow X_j \neq k \quad \forall i \neq j$$
$$X_i = k_i, X_j = k_j \Longrightarrow |i - j| \neq |k_i - k_j|$$



Example: Map coloring

- Variables: {WA, NT, SA, Q, NSW, V, T}
- Domains: {red, blue, green}
- Constraints: adjacent regions have different colors
  - Implicit: WA ≠ NT, WA ≠ SA, SA ≠ NT, NT ≠ Q,…
  - Explicit: (WA,NT) ∈ {(red,green), (red,blue),…}

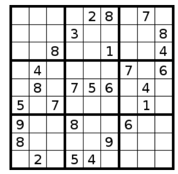


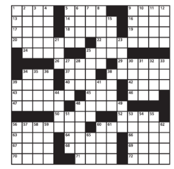
Example: Task scheduling

- Variables: {AxleF, AxleB, WheelRF, WheelLF,…,Inspect}
- Domains: Time task starts D = [0, 1, 2,…, ∞)
- Constraints:
  - Axle must be done before the wheel
    - AxleF + 10 < WheelLF
    - AxleF + 10 < WheelRF
  - The front axle and the back axle cannot be done at the same time
    - (AxleF + 10 < AxleB) OR (AxleB + 10 < AxleF)
  - Everything must be done within 30 minutes
    - Change domains to have upper bound 30 min.

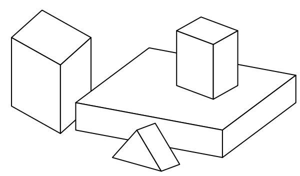
More examples

- More toy examples
  - sudoku, cryptarithmetic





- Real-world applications
  - Interpreting lines in 3D
  - Assignment problems, e.g. who teaches what class?
  - Timetable problems, e.g. which class offered when? where?
  - Transportation scheduling
  - Factory scheduling
  - Circuit layout



Types of CSPs - variables

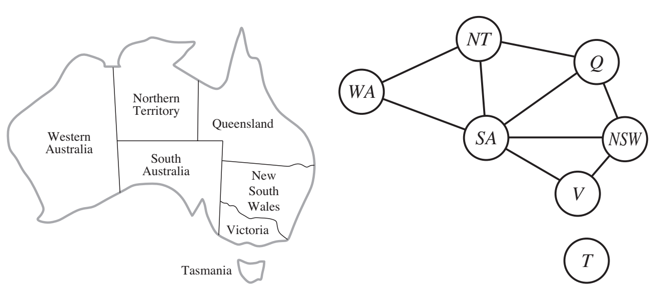
- Discrete variables
  - Finite domains
    - size d means $O(d^n)$ possible assignments to explore
  - Infinite domains
    - Linear constraints (e.g. $T_1 + d_1 \leq T_2$) are solvable
    - Non-linear constraints undecidable
- Continuous variables
  - linear programming problems with linear equality or inequality constraints solvable in polynomial time

Types of CSPs - constraints

- Unary constraints involve a single variable
  - e.g. SA ≠ green
- Binary constraints involve pairs of variables
  - SA ≠ NSW
  - A binary CSP can be illustrated using a constraint graph
- Higher-order constraints
  - e.g. A, B, and C cannot be in the same grouping
  - e.g. AllDiff (all variables must be assigned different values)
- Preference constraints
  - costs on individual variable assignments
  - constraint optimization problem

Constraint Graph

- Useful for binary constraint CSPs where each constraint relates (at most) two variables
- Nodes correspond to variables
- Edges (arcs) link two variables that participate in a constraint
- Use graph to speed up search



Inference: constraint propagation

- Use the constraints to reduce the number of legal values for a variable
- Possible to find a solution without searching
  - Node consistency
    - A node is node-consistent if all values in its domain satisfy the unary constraints
  - Arc consistency
    - A node X_i is arc-consistent w.r.t. node X_j if for every value in D_i there exists a value in D_j that satisfies the binary constraint
    - Algorithm AC-3
  - Other types of consistency (path consistency, k-consistency, global constraints)

AC-3 algorithm for Arc consistency

```
function AC-3(csp) returns false if inconsistency found, true otherwise
    queue ⟵ all arcs in csp
    while queue not empty
        (Xi,Xj) ⟵ REMOVE-FIRST(queue)
        if REMOVE-INCONSISTENT-VALUES(Xi, Xj)
            if size Di == 0 return false
            for each arc (Xk ,Xi)
                add (Xk, Xi) to queue
    return true
```

```
function REMOVE-INCONSISTENT-VALUES(Xi, Xj)
    revised ⟵ false
    for each x in Di
        if ∄ y in Dj s.t. (x,y) satisfies constraints
            delete x from Di
            revised ⟵ true
    return revised
```

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c constraints (arcs)
d domain size

O(cd)

Total $O(cd^3)$

```
function REMOVE-INCONSISTENT-VALUES(Xi, Xj)
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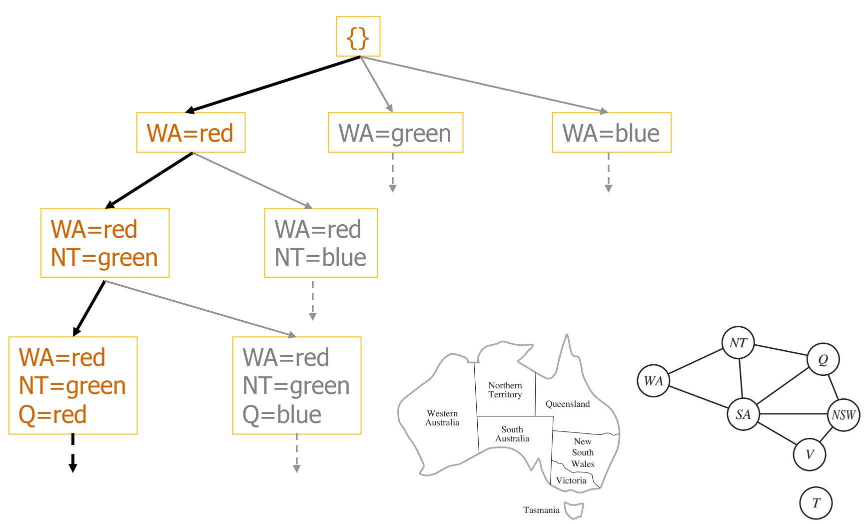
$O(d^2)$

Search



Backtracking Search

```
function CSP-BACKTRACKING(assignment) returns a solution or failure
    if assignment complete return assignment
    X ⟵ select unassigned variable
    D ⟵ select an ordering for the domain of X
    for each value in D
        if value is consistent with assignment
            add (X = value) to assignment
            (ADD INFERENCE HERE)
            result ⟵ CSP-BACKTRACKING(assignment)
            if result ≠ failure return result
        remove (X = value) from assignment
    return failure
```

Backtracking Search

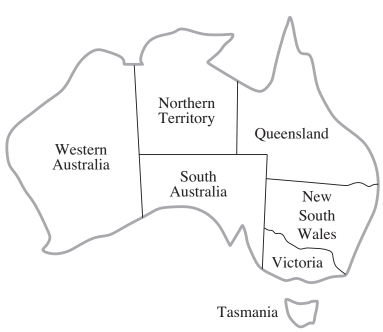
- Backtracking search is the basic uninformed algorithm for solving CSPs
- Idea 1: One variable at a time
  - Variable assignments are commutative so fix ordering
  - i.e. (WA = red, NT = green) is the same as (NT = green, WA = red)
- Idea 2: Check constraints as we go
  - Consider only values which do not conflict with previous assignments
  - May take some computation to check
  - "incremental goal test"
  - (Additional inference is optional, e.g. arc-consistency)
- Depth-first search with these 2 improvements is called *backtracking search*

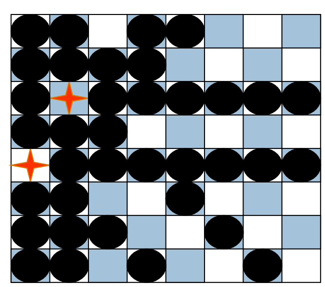
Improving backtracking search

- Ordering
  - Which variable X should be assigned a value next?
  - In which order should its domain D be sorted?
- Incorporating inference
  - Can we detect inevitable failure early?
- Structure
  - Can we exploit the problem structure?

Ordering

- Which variable should we choose?





Ordering

- Variable ordering
  - Minimum-remaining values heuristic - Choose the variable with the fewest "legal" moves remaining
  - Degree heuristic - Choose variable involved in the largest number of constraints with remaining unassigned variables
- Value ordering
  - Least-constraining value heuristic - Choose the value that rules out the fewest choices for the neighboring variables

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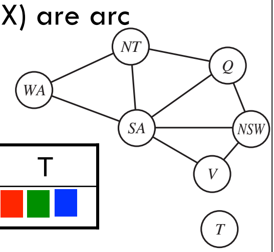
Incorporating inference: forward checking

- After an assignment X = x, look at each unassigned variable Y connected to X by a constraint
  - Delete from Y's domain any value inconsistent with X = x
  - Equivalent to ensuring all arcs of the form (Y, X) are arc consistent



WA	NT	Q	NSW	V	SA	T
Red Green Blue	Red Green Blue	Red Green Blue	Red Green Blue	Red Green Blue	Red Green Blue	Red Green Blue

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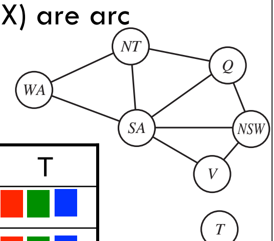

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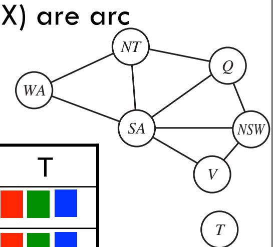

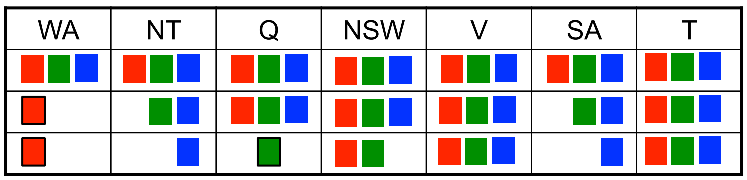

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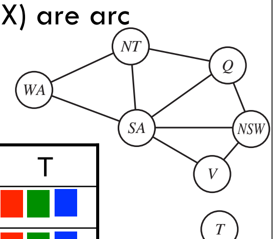

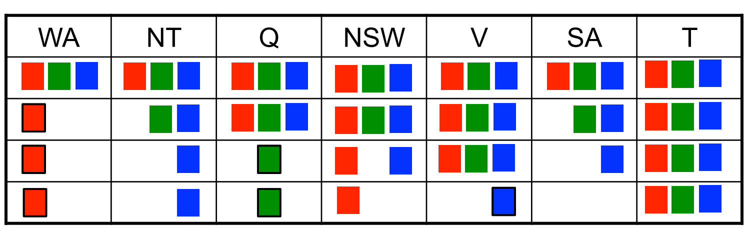

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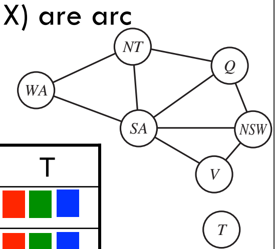
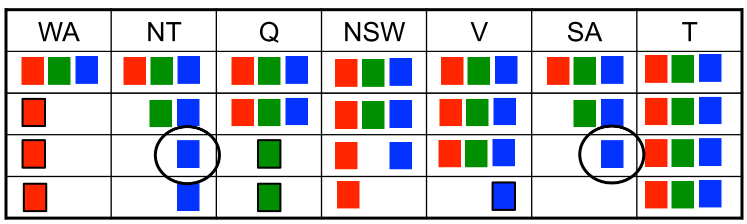
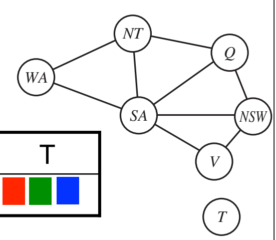


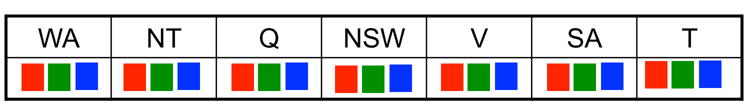


Could have detected earlier that things were going wrong!

Incorporating inference: AC-3

- Can also infer path consistency based on triples of variables and k-consistency





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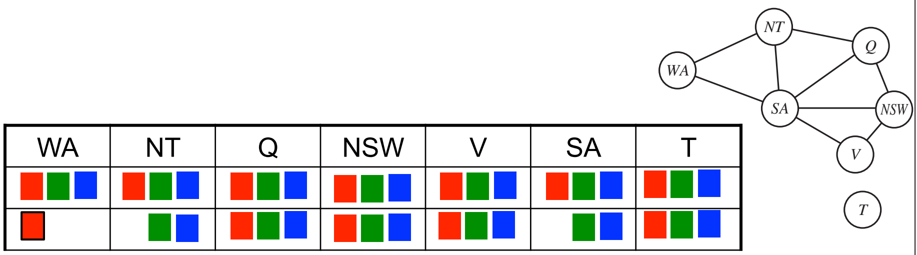

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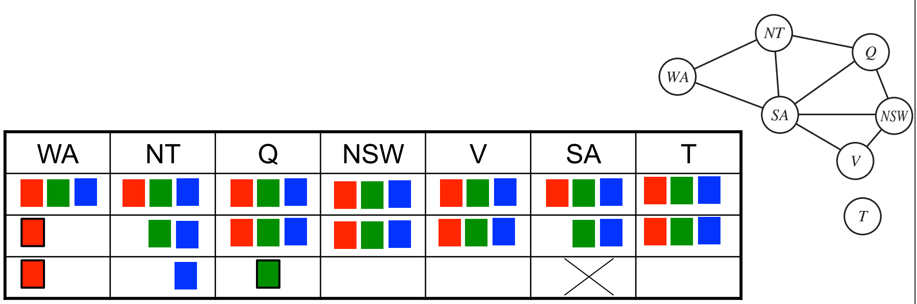

Improving backtracking search

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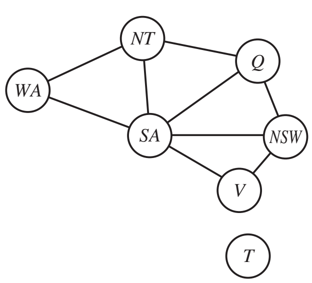
Structure of the constraint graph

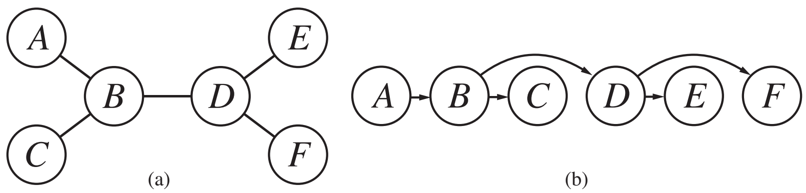
- Independent subproblems
  - Find connected components of the constraint graph
  - e.g. Tasmania and the mainland are independent
  - If we can split n variables into c subproblems of n/c variables each: $O(d^n) \longrightarrow O(d^c\, n/c)$



Tasmania is independent of the mainland!

Structure of the constraint graph

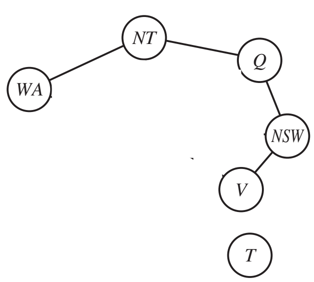
- Tree structured constraint graphs
  - Can solve in linear time using AC-3

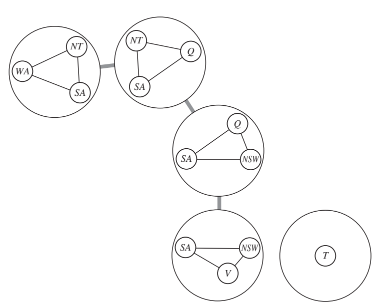


(a) (b)

Structure of the constraint graph

- Reduction to a tree structured graph
  - Cycle cutset – a subset of the variables whose removal creates a tree.
  - Tree decomposition – Divide graph into subproblems, solve independently merge the solutions

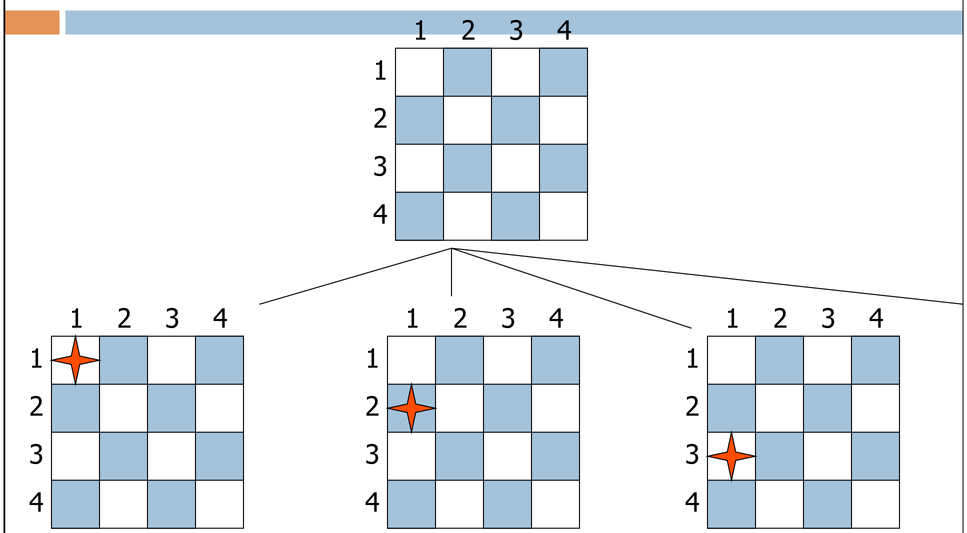




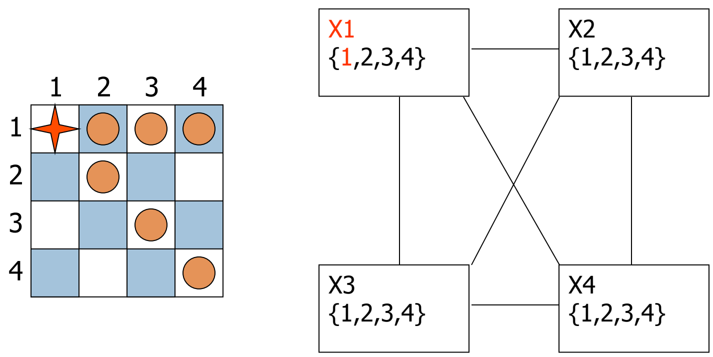
CSP Summary

- Constraint Satisfaction Problems (CSPs)
- Solving CSPs using inference
- Solving CSPs using search
  - Backtracking algorithm = DSF + constraints checking
  - General (not problem-specific) heuristics
- Improving Backtracking
  - Variable and value ordering
  - Inference
  - Structure

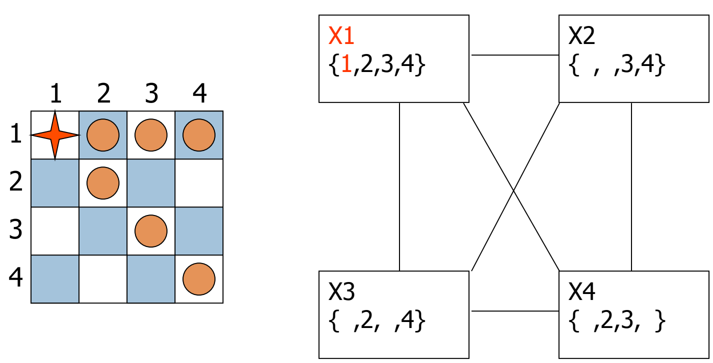
4-Queens Problem



4-Queens Problem



4-Queens Problem



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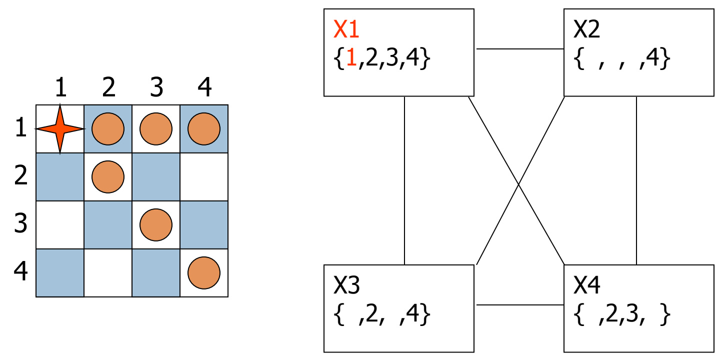

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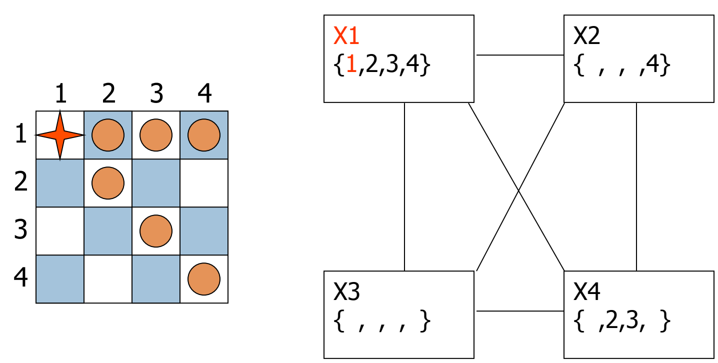

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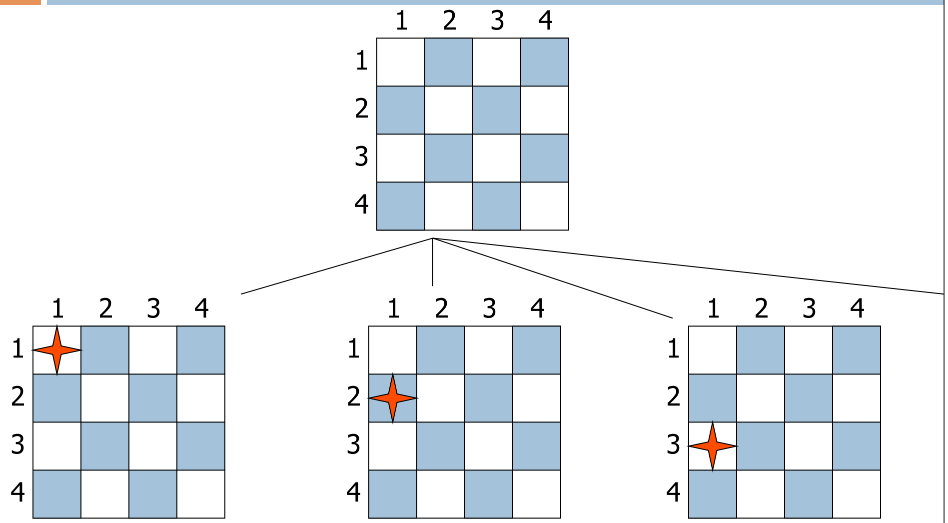

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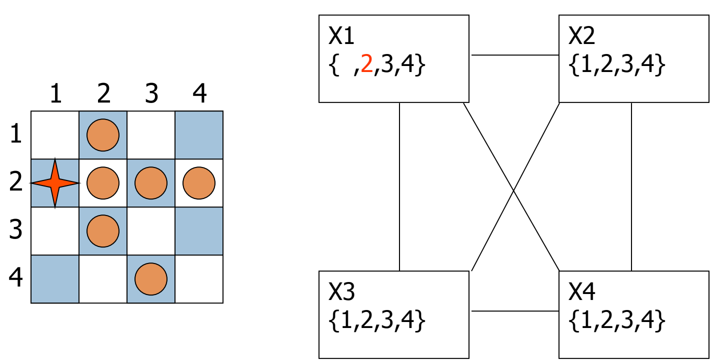

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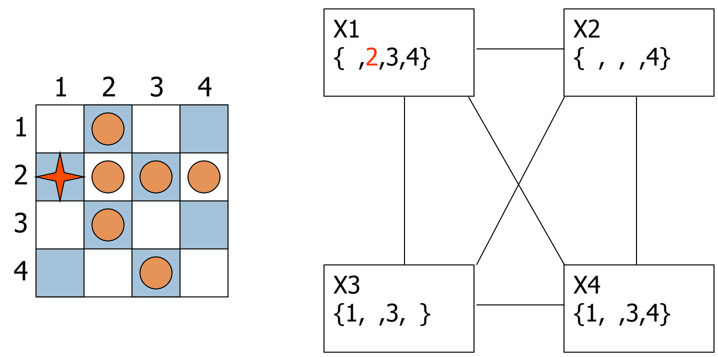

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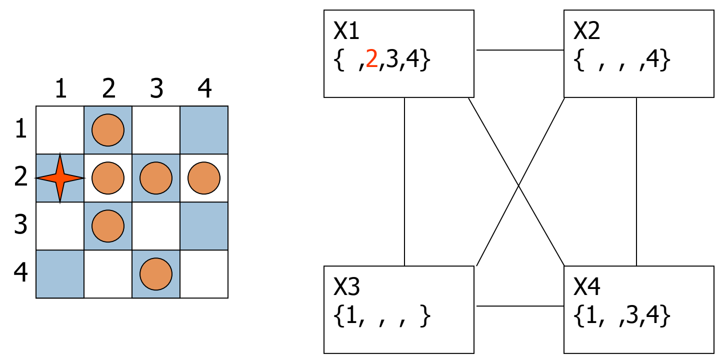

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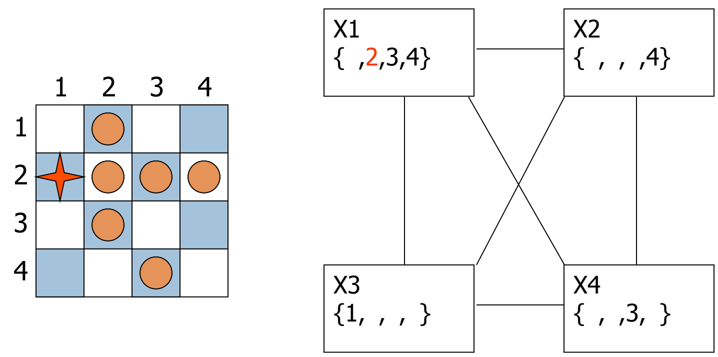

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